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COMPUTING AERODYNAMIC SOUND USING ADVANCED STATISTICAL TURBULENCE THEORIES

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SUMMARY

The Lighthill theory of aerodynamic sound requires a knowledge of the spatial and temporal variation of the two-point, two-time turbulent velocity correlations. The feasibility of determining these correlations based on extending closure models for one-point, one-time turbulence correlations is demonstrated. The procedure is based on a spatial moment integral formulation of the governing equations using approximate, parameterized trial functions for the two-point, two-time velocity correlations. Solution of the equations results in a set of anisotropic length scales and the separation-time-dependent decorrelation of the ensemble averaged turbulent velocities. The analysis was simplified using the assumption of homogeneous stationary turbulence and a constant shear, unidirectional mean flow.

It is shown that the anisotropic behavior of measured turbulence correlations can be characterized by this technique. Using the Proudman formulation of the Lighthill integral and the assumption of normal joint probability, measured sound power directivity can be reproduced for the compact acoustic limit by assigning a specific separation-time behavior to a decorrelation function (which becomes the viscous dissipation in the limit of zero separation).

It is concluded that the present approach is a viable technique for the prediction of turbulence generated aerodynamic noise. It is recommended that further effort should be concentrated on extending the theory to noncompact sound generation, developing a theory for the behavior of the decorrelation function, and to investigate more fully the effect of the anisotropic scales on sound generation.

1. INTRODUCTION

The theory of aerodynamic sound generation by turbulent flows, under certain simplifying assumptions, is based upon the availability, either experimentally or theoretically, of two-point, two-time Eulerian velocity correlations. Theoretical analyses of the sound generation problem have generally adopted an isotropic form for these correlations.

In addition to the form of the spatial and temporal variation of the velocity correlations, accurate predictions of sound generation are dependent upon an ability to specify the one-point, one-time limit. The success of modeling efforts to close the turbulent rate equations for velocity correlations at second-order has provided the means to predict these one-point, one-time correlations with greater confidence than previously. Bilanin and Hirsh[1] have used such a model to predict the sound radiated from a turbulent swirling jet. Since only the one-point, one-time behavior of the velocity correlations could be predicted by their method, it was still necessary to adopt an assumed form for the characteristics of the velocity correlations in separation space and time. For this. the Ribner[2] formulation was used.

The feasibility of developing a more general representation of the two-point, two-time turbulent velocity correlations is explored in this study. If a Gaussian joint probability density function is assumed, the turbulence statistics required to predict aerodynamic sound are the two-point, two-time turbulent velocity correlations $Q_{ij}(\vec{x},\vec{\xi},t,\tau)$ where $\vec{x}$ and t are absolute space and time and ξ and τ are the separation in space and time. $Q_{ij}(\vec{x},\vec{\xi},t,\tau)$ is defined by

$$Q_{ij}(\vec{x},\vec{\xi},t,\tau) = \overline{u_i(\vec{x},t)u_j'(\vec{x} + \vec{\xi},t + \tau)}$$

where the bar denotes ensemble time average. If the turbulence is assumed to be homogeneous and stationary, the velocity correlation functions are dependent only upon separation space and time.

The equations governing $Q_{ij}(\vec{x},\vec{\xi},t,\tau)$ contain higher order correlations in two-point, two-time triple velocity correlations, as well as correlations between turbulent pressure and velocity fluctuations. The closure problem is again the fundamental obstacle to the solution of the equations, as it is in the determination of single-point, single-time turbulent correlations. In recent years, however, methods of modeling the higher order terms in the one-point, one-time problem have been developed, resulting in significant improvement in zero separation turbulent predictions compared to previous mixing length models, as pointed out above. Successful application of these modeling techniques to the zero separation problem suggests that they may be extended to encompass two-point

correlations in space and possibly in time.

Such an application presents numerous difficulties. The additional variable in relative time-separation as well as relative spatial separation increase the difficulties in the solution to the governing equations many fold. Even this is no fundamental barrier to obtaining solutions, however. Higher order closure is again the basic problem, compounded by the additional spatial and temporal independent variables.

In this study a new approach to the two-point, two-time turbulence problem has been attempted. Certain simplifying assumptions have been made to reduce the complexity of the problem for this initial study since testing the feasibility of the approach is the first priority. Therefore the turbulence is assumed to be homogeneous and stationary everywhere; conditions which later can be relaxed fairly simply in the absolute space variables under certain conditions. The governing differential equations are simplified by assuming a constant unidirectional mean shear flow. The complexity of the equations is finally reduced to manageable proportions by employing an integral technique. Approximate forms of the velocity correlations are chosen which are capable of reproducing the main features of experimentally observed correlations, each containing a set of separation-time-dependent parameters which are essentially the anisotropic scaling factors. These approximate forms are substituted into the governing momentum equations and a family of moments in powers of the separation coordinates is taken over all space. The result is a set of simultaneous separation-time-dependent ordinary differential equations which may be solved for the time variation of the scale parameters. A set of subsidiary equations or constraints are employed which are derived from an integral form of the continuity equations.

Once the scale parameters, as well as their time dependence, are found, enough information is available to determine the aerodynamic sound associated with this type of turbulent flow. Since the source term for the sound power is proportional to the fourth derivative of separation time, a method must be devised to determine the fourth derivatives of the scale parameters. In this initial study we examine the compact limit of sound generation and the theory will be applied at zero separation time. Thus, we avoid the necessity of integrating the governing differential equations and instead expand the parameters in powers of τ , the separation time, and determine the coefficients of these expansions for small τ .

Although a number of simplifications have been introduced into the analysis during this initial study, they may be removed with varying amounts of difficulty once the feasibility of the technique is demonstrated. Adoption of a working hypothesis that under special circumstances the turbulent behavior of a flow can be inferred from the analysis of simpler flows reduces this difficulty substantially.

For instance, if the characteristic size of the biggest eddies is smaller than the characteristic length of the flow over which the mean and turbulent variables may be considered homogeneous suggests that we may assume the flow to be homogeneous locally. Even if this condition is not precisely met, the homogeneous solution may serve as the first approximation to the actual solution. This point of view is adopted in this analysis. Thus, for instance, we shall assume that the velocity gradient is constant, and the solution for a varying gradient is obtained by a local application of the present analysis.

This report is organized as follows:

Section 2 is a synoptic of the approach taken.

In Section 3 the equation defining the sound power intensity is introduced.

In Section 4 the equations governing the two-point, two-time velocity correlations are derived.

In Section 5 the models to be used for the pressure-velocity triple-velocity, and dissipation terms are developed.

Section 6 summarizes the one-point, one-time turbulence closure model used to determine the zero-separation velocity correlations which are the limit for the two-point correlations.

In 7 the correlation function required to represent the approximate spatial and temporal variations of Q_{ij} is selected. This is introduced into the governing equations in Section 8 and the appropriate equations are obtained.

In 9 a qualitative comparison between theoretical results and experimental data is presented.

In Section 10 the model is incorporated into the sound power integral and theoretical results calculated for an annular shear layer are compared with acoustic intensity data taken with an axisymmetric jet in Section 11.

In Section 12, conclusions and recommendations are offered.

NOMENCLATURE

A	proportionality constant relating turbulent intensity to integral scale and shear, Eq. (6.7)
A_{ij}, B_{ijpq}	directional factors in azimuthal (ϕ) and latitudinal (ψ) direction for sound power, Tables 5a and 5b
$\overline{A}_{ij}, \overline{B}_{ijpq}$	directional factors averaged in ϕ-direction in axisymmetric flow, Tables 6a and 6b

a	low Reynolds number constant in isotropic microscale equation, Eq. (5.6)
b	dissipation factor at high Reynolds number at zero separation; decorrelation parameter averaged over space for two-point, two-time viscous effects
$C_{ij}^{(m_k)}$	integral moments defined by Eq. (8.7)
c_k	scale factor in k^{th} direction
c_o	ambient speed of sound
$D_{ij}^{(m_k)}$	integral moments defined by Eq. (8.8)
G_2, G_4	series expansion functions defined by Eq. (9.13)
$g_{ij}(\tau)$	memory function for ij correlation
$I_{ij}^{(m_k)}$	integral moments defined by Eq. (8.6)
$I(\psi)$	total sound power intensity for axisymmetric flow per unit area in ψ direction at observer distance x
L_m	reference length
N	v_c/A
P	mean pressure
p	fluctuating pressure
P_{ij}	pressure-velocity correlations, defined by Eq. (4.19)
$P(\psi,\phi,\vec{y})$	sound power intensity per unit volume
$P(\psi,\vec{y})$	ϕ average of sound power intensity per unit of volume
$\hat{P}(\psi,y)$	$c_o^5 x^2 L_m P(\psi,\phi,y)/(P_o U_m^8)$
q	turbulent intensity
Q_{ij}	two-point, two-time turbulent velocity correlation
R_{ij}	defined by Eq. (8.4)
$\vec{r}$	vectorial distance from origin of separation coordinates
R_J	axisymmetric jet radius to middle of annular shear layer
S_{ij}	triple-velocity correlations, defined by Eq. (4.20)

s_k	$\frac{1}{2}(x_k + x_k')$, absolute space coordinate
T_{ij}	quadrupole strength density, Eq. (3.2)
t	absolute time
U_k	mean velocity in k^{th} direction
U_{1_o}	mean velocity in x_1 direction at $\xi_k = 0$
u_k	fluctuating velocity in k^{th} direction
$(u_i u_j)_o$	one-point, one-time turbulent velocity correlation
u_{ij}	$(u_i u_j)_o / q^2$
V_F	integration volume
v_c	triple-velocity correlation modeling constant
v_k	mean plus fluctuating velocity in k^{th} direction
v_x	total velocity in direction of $\vec{x}$
W_{ij}, W_{ijpq}	defined by Eq. (10.7)
X_{ij}, Y_{ij}	defined by Eq. (10.13f) and Eq. (10.13g)
x_k	absolute coordinate in k^{th} direction
$\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \mu_{ij}$	correlation function separation time dependent scale parameters
$\hat{\alpha}_{ij}, \hat{\beta}_{ij}, \hat{\gamma}_{ij}, \hat{\mu}_{ij}$	$\alpha_{ij}\sigma_{ij}^2 c_1^2$, $\beta_{ij}\sigma_{ij}^2 c_2^2$, $\gamma_{ij}\sigma_{ij}^2 c_3^2$, $\mu_{ij}\sigma_{ij}^2 c_1 c_2$
δ_{ij}	Kronecker delta
Δ_s	shear layer thickness
ε_{ij}	dissipation of ij correlation
η_{ij}	defined by Eq. (4.5)
κ_{ijpq}	defined by Eq. (10.11)
$\hat{\kappa}_{ij}, \hat{\kappa}_{ijpq}$	defined by Eq. (10.13c), Eq. (10.13d) and Eq. (10.13e)

Λ	isotropic form of integral length scale
$\Lambda_{ij}^{(k)}$	anisotropic integral length scale of ij correlation in k^{th} direction
λ	turbulent microscale
μ	viscosity
ν	kinematic viscosity
ξ_k	separation coordinate; $(x_k - x_k')$
$\Pi(m_k)$	moment function defined by Eq. (8.5)
ρ, ρ_o	density, ambient density
σ_{ij}	decay length scale of ij correlation
τ_{ij}	viscous compressive stress tensor
τ	separation time
ϕ, ψ	angular spherical coordinates

Superscripts:

$(\vec{\ })$	vector quantity
$(\ )'$	quantity at x' and t'
$(\ )''$	quantity at x'' and t''
$(\bar{\ })$	ensemble average

Subscripts:

$(\ )_m$	reference quantity
$(\ )_n$	index of coefficients in Taylor series expansion
$(\ )_k$	value in k^{th} direction

2. SYNOPTIC OF THEORY

The length and complexity of the theoretical turbulence and acoustic developments contained in the following pages make it desirable to summarize the approach more fully at this point. Lacking this, there is some danger that the objectives of the present study, and its important lines of approach, might be lost in the details of the analysis.

2.1. Sound Intensity

The Lighthill theory of aerodynamic sound[3,4] requires a knowledge of the spatial and temporal variation of the two-point, two-time turbulent ensemble averaged velocity correlations $Q_{ij}(\vec{x},\vec{\xi},t,\tau)$. The sound intensity generated by the velocity fluctuation and radiated in the spherical coordinate direction (ψ , ϕ), at distance x from the source, is

$$I(\psi,\phi) = \frac{\rho_o}{16\pi^2 c_o^5} \iiint_\infty \frac{d\vec{x}}{x^2} \iiint_\infty \frac{\partial}{\partial \tau^4} \overline{v_x^2 v_x'^2}\, d\vec{\xi} \tag{2.1}$$

$v_x(\vec{x},t)$ and $v_x'(\vec{x} + \vec{\xi},t + \tau)$ are the velocity components at $(\vec{x},t)$ and $(\vec{x} + \vec{\xi},t + \tau)$ in the direction of the radiated power and contain the mean plus fluctuating turbulent velocities. The assumption of a normal joint probability distribution permits expressing $\overline{v_x^2 v_x'^2}$ in terms of Q_{ik} , $Q_{ij}Q_{pq}$ and $Q_{iq}Q_{pj}$, plus other noncontributing terms. Obtaining Q_{ij} is the objective of the turbulence analysis.

2.2. Governing Equations for Q_{ij}

Developing the governing equations for Q_{ij} yields a set of differential equations having higher order correlations $\overline{u_i u_j u_k'}$ and pressure velocity cross-correlations, $\overline{u_i p'}$. Equations for the third-order terms contain terms in fourth-order velocity correlations. This is the so-called closure problem where equations for any order include correlations one order higher. An infinite hierarchy of equations result, each level of which does not contain sufficient information to obtain a solution.

This problem is avoided by modeling the third-order and pressure-velocity correlations in terms of second-order correlations. Extension of an established second-order closure technique is used for this, yielding a consistent set of (solvable) equations for Q_{ij} .

The equations governing the two-point, two-time correlations, Q_{ij} , which are derived in this study, are

$$2\frac{\partial Q_{ij}}{\partial t} + \frac{\partial Q_{ij}}{\partial \tau} + Q_{jk}\frac{\partial U_i}{\partial x_k} + Q_{ik}\frac{\partial U_j'}{\partial x_k'} + \frac{1}{2}[U_k + U_k']\frac{\partial Q_{ij}}{\partial s_k}$$

$$+ [U_k - U_k']\frac{\partial Q_{ij}}{\partial \xi_k} = -\frac{1}{2}\frac{\partial}{\partial s_k}[\overline{u_i u_j' u_k'} + \overline{u_i u_k u_j'}]$$

$$- \frac{\partial}{\partial \xi_k}[\overline{u_i u_j' u_k'} - \overline{u_i u_k u_j'}] - \frac{1}{2\rho}\left[\frac{\partial}{\partial s_i}\overline{u_j' p} + \frac{\partial}{\partial s_j}\overline{u_i p'}\right]$$

$$- \frac{1}{\rho}\left[\frac{\partial}{\partial \xi_j}\overline{u_i p'} - \frac{\partial}{\partial \xi_i}\overline{u_j' p}\right] + \frac{1}{2}\nu\frac{\partial^2 Q_{ij}}{\partial s_k^2} + 2\nu\frac{\partial^2 Q_{ij}}{\partial \xi_k^2} \tag{2.2}$$

where

$$\xi_k = x_k' - x_k \tag{2.3a}$$

$$s_k = \frac{1}{2}[x_k + x_k'] \tag{2.3b}$$

$$\tau = t' - t \tag{2.3c}$$

Assuming that the turbulence is homogeneous and steady eliminates derivatives in s_k and t and assumption of a constant, unidirectional mean velocity gradient in the x_2 direction

$$U_k = \delta_{1k}\left[U_{1_o} + \xi_2\frac{dU_1}{dx_2}\right] \tag{2.4}$$

affords considerable simplification. The following turbulent models are then introduced. For the triple-velocity correlations

$$\frac{\partial}{\partial \xi_k}[\overline{u_i u_j' u_k'} - \overline{u_i u_k u_j'}] = -v_c q \Lambda \nabla^2 Q_{ij} \tag{2.5}$$

The pressure velocity correlations are modeled as Rotta tendency-towards-isotropy terms

$$\frac{1}{\rho}\left[\frac{\partial}{\partial \xi_j}\overline{u_i p'} - \frac{\partial}{\partial \xi_i}\overline{u_j' p}\right] = \frac{q}{\Lambda}\left[Q_{ij} - \frac{1}{3}\delta_{ij}Q_{\ell\ell}\right] \qquad (2.6)$$

The dissipation model (high Reynolds number limit) is

$$2\nu\nabla^2 Q_{ij} = -2b\,\frac{q}{\Lambda}\,Q_{ij} \qquad (2.7)$$

Here Λ is a global macroscale set by the local flow and $q^2 = \overline{u_i u_i}$. The equations then take the form

$$\frac{\partial Q_{ij}}{\partial \tau} + \left(U_{1_o} + \xi_2\frac{\partial U_1}{\partial x_2}\right)\frac{\partial Q_{ij}}{\partial \xi_1} = -\left(\delta_{i1}Q_{2j} + \delta_{j1}Q_{i2}\right)\frac{dU_1}{dx_2}$$

$$- \frac{q}{\Lambda}\left[Q_{ij} - \frac{1}{3}\delta_{ij}Q_{\ell\ell}\right] + v_c q\Lambda\nabla^2 Q_{ij} - \frac{2bq}{\Lambda}\,Q_{ij} \qquad (2.8)$$

b and v_c are modeling constants. The form of the modeled terms are adapted from Donaldson's second-order closure technique.[5]

2.3. One-Point, One-Time Correlations

The limit of Q_{ij} for ξ and τ equal to zero are the one-point,one-time velocity correlations. The equations governing $Q_{ij}\underline{(0,0)}$ may be solved to determine the one-point,one-time values of $\overline{u_i u_j}$ in a constant shear flow. The results are

$$\overline{u_1 u_1}/q^2 = \frac{1}{3}[1 + 4b] \qquad (2.9a)$$

$$\overline{u_2 u_2}/q^2 = \frac{\overline{u_3 u_3}}{q^2} = \frac{1}{3}[1 - 2b] \qquad (2.9b)$$

$$\overline{u_1 u_2}/q^2 = \frac{1}{3A}\,[1 - 2b] \qquad (2.9c)$$

$$A = \left[\frac{1 - 2b}{3b}\right]^{\frac{1}{2}} \qquad (2.9d)$$

where $A = q/(\Lambda dU_1/dx_2)$.

The one-point, one-time solution used in this study is for convenience only. The one-point, one-time correlation can be obtained from any turbulence model, or from numerical solutions of shear flow turbulence.

2.4. Two-Point, Two-Time Correlation Functions

This study employs an integral approach to determine the behavior of $Q_{ij}(\xi_k,\tau)$. An analytical form is selected for Q_{ij} which is capable of approximating the two-point, two-time behavior of velocity correlations seen in available experimental data. This form contains a set of separation-time-dependent free parameters which are determined using the governing differential equations together with an integral approach. Here, we have chosen the following form to approximate Q_{ij}

$$Q_{ij} = u_{ij} R_{ij} g_{ij}(\tau)$$

$$R_{ij} = [1 - \alpha_{ij}(\xi_1 - U_1\tau) - \beta_{ij}\xi_2^2 - \gamma_{ij}\xi_3^2 - \mu_{ij}(\xi_1 - U_1\tau)\ \xi_2]$$

$$\cdot \exp -\left[\frac{1}{\sigma_{ij}{}^2}\left\{\frac{1}{c_1^2}\ (\xi_1 - \quad \tau)^2 + \frac{\xi_2^2}{c_2^2} + \frac{\xi_3^2}{c_3^2}\right\}\right] \tag{2.10}$$

where $\overline{u_i u_j}$ is the zero separation, one-point, one-time result. R_{ij} contains the parameters α_{ij} , β_{ij} , γ_{ij} , μ_{ij} , σ_{ij} , c_1 , c_2 and c_3 which are separation time dependent. α_{ij} , β_{ij} , γ_{ij} and μ_{ij} serve to provide zero crossings of the correlations, the anisotropic analog of the isotropic velocity correlation function $g(r)$ zero crossing. The asymmetry in the ξ_2 direction due to shear is accounted for by the presence of μ_{ij} . c_1 , c_2 and c_3 are anisotropic scaling functions to provide variations in spread in the three coordinate directions. $g_{ij}(\tau)$ is a memory function which decorrelates Q_{ij} at $\xi_k = 0$ in a coordinate system following the flow at velocity U_{1_o} . In order to evaluate the anisotropic behavior of the turbulent scales, we define

$$\Lambda_{ij}^{(k)}(\tau) = g_{ij}(\tau)\int_0^\infty R_{ij}(\xi_k - \delta_{1k}U_{1_o}\tau)d(\xi_k - \delta_{1k}U_{1_o}\tau)\ , \ k = 1,2,3 \tag{2.11}$$

which becomes a function of the correlation function parameters.

2.5. Continuity

The continuity equations governing $Q_{ij}(\xi_k,\tau)$ are

$$\frac{\partial Q_{ij}}{\partial \xi_j} = 0 \tag{2.12}$$

These equations serve to add constraints on the parameters and provide continuity in an integral sense as will be demonstrated shortly.

2.6. Integral Moments

The correlation functions are substituted into the governing equations and a set of integral moments of the equations are taken over all space. The moment function

$$I_{ij}^{(m_k)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Pi(m_k) Q_{ij}\; d\xi_1 d\xi_2 d\xi_3 \tag{2.13}$$

is defined where $\pi(m_k)$ is a function of the ξ_k coordinates

$$\Pi(m_1,m_2,m_3) = \xi_1^{m_1}\xi_2^{m_2}\xi_3^{m_3} \tag{2.14}$$

A sufficient number of moments are taken to provide enough equations, together with the continuity constraints, to allow solution of the set for all the unknown free parameters. The governing equations become total differential equations in τ in the separation time dependent parameters. Since experimental data indicates that $\partial Q_{ij}/\partial\tau = 0$ at zero separation time, the initial conditions are found by an algebraic solution of the equation set for the parameters, with zero rate of change at $\tau = 0$. The characteristic length of the flow is specified by independent selection of one scale of the set $\Lambda_{ij}^{(k)}$.

Continuity is satisfied in an integral sense by integrating the continuity equation over half the separation space domain as follows

$$\int_{-\infty}^{\infty} d\xi_p \int_{-\infty}^{\infty} d\xi_q \int_{-\infty}^{0} \frac{\partial Q_{ij}}{\partial \xi_j}\, d\xi_r = 0 \quad ; \quad pqr = 123,231,312 \tag{2.15}$$

2.7. Decorrelation Function $b(\tau)$

Decorrelation of Q_{ij} with separation time is provided for in this study by a separation-time dependence of the turbulent dissipation function $b(\tau)$. At zero separation in space and time, b is a modeling constant. The one-point, one-time value of $b = 1/8$ is probably no longer adequate when considering a space integrated dissipation of Q_{ij}. Instead a time varying form $b(\tau)$ is hypothesized. $b(0)$ and the time dependence is selected to provide agreement between the theoretically calculated sound generation and measured sound intensity data.

2.8. Solution of Time Dependent Equations

The resulting equation set may be solved numerically. However, for this first effort the compact acoustic limit is assumed. This permits expansion of the parameters in power series in τ about $\tau = 0$. The zero-order solution is the initial condition now known. Higher-order coefficients are found by substitution of the expansions in the differential equations and ordering of terms. The result is a set of algebraic equations for coefficients of order $n + 1$ in terms of coefficients of order n. The decorrelation function $b(\tau)$ is specified by selection of $b(0)$ and the higher order coefficients to provide agreement between measured and calculated sound intensity distributions.

2.9. Calculation of Sound Intensity

Having the behavior of Q_{ij} for the selected shear flow considered, the results can be used to evaluate $\overline{v_x^2 v_x'^2}$ $(\overline{u_i u_j}$, $\frac{dU_1}{dx_2}$, x_1, x_2, x_3, ψ, $\phi)$ and the fourth-derivative is then taken with respect to τ. Then τ is set equal to zero and the integration taken over separation space. Since Q_{ij} is an explicit function of ξ_k this is done analytically. We thus have the compact limit of the locally radiated sound intensity, which is a function of the local gradient dU_1/dx_2 and the one-point turbulence correlations $\overline{u_i u_j}$. A final numerical integration over a specified mean velocity profile and turbulence distribution provides the sound intensity radiated in the (ψ,ϕ) direction.

This has been a very brief overview of the theoretical approach. The analysis is presented in considerably more depth in Sections 3 to 11.

3.0 ACOUSTIC SOURCE TERM MODEL

The sound pressure radiated to a point x in the far field in a localized unsteady or turbulent flow was shown by Lighthill to be given by

$$p(\vec{x},t) = \frac{x_i x_j}{4\pi c_o^2 x^3} \int_{\infty} \left[\frac{\partial^2 T_{ij}}{\partial t^2}\right] d\vec{y} \tag{3.1}$$

T_{ij} is a quadrupole strength density

$$T_{ij} = \rho v_i v_j + \tau_{ij} + (P - c_o^2 \rho)\,\delta_{ij} \tag{3.2}$$

where $\rho v_i v_j$ is the unsteady momentum flux, v_i the velocity, τ_{ij} the viscous compressive stress tensor, P the local pressure, c_o the ambient speed of sound, ρ the density, and δ_{ij} the Kronecker delta. Equation (3.2) is normally dominated by the unsteady momentum flux $\rho v_i v_i$. The symbol [ ] in Eq. (3.1) denotes retarded time. The indices i, j and k are equal to 1, 2 or 3 and repeated indices are summed. Figure 1 presents a schematic diagram of the coordinates employed here.

The sound power generated by the velocity fluctuations and radiated in the directions (ψ,ϕ) in spherical coordinates is the ensemble averaged pressure fluctuations at the point of observation divided by $\rho_o c_o$. Defining this by $I(\psi,\phi)$

$$I(\psi,\phi) = \frac{x_i x_j x_k x_l}{16\pi^2 \rho_o c_o^5 x^6} \iint_{\infty\,\infty} \overline{\frac{\partial^2(\rho' v_i' v_j')}{\partial t^2}\,\frac{\partial^2(\rho'' v_k'' v_l'')}{\partial t^2}}\, d\vec{y}'\, d\vec{y}'' \tag{3.3}$$

where the first term of the integrand is evaluated at $\vec{y}', t'$ and the second at $\vec{y}'', t''$ and the bar denotes an ensemble average. $t' - t''$ is the difference in time of travel to x from y' and $\vec{y}''$. Ribner[6] expressed Eq. (3.3) as a function of the midpoint $\vec{y}$ and the separation in space and time using

$$\vec{y} = \tfrac{1}{2}(\vec{y}' + \vec{y}'') \tag{3.4a}$$

$$\vec{\xi} = \vec{y}' - \vec{y}'' \tag{3.4b}$$

$$\tau = t' - t'' \tag{3.4c}$$

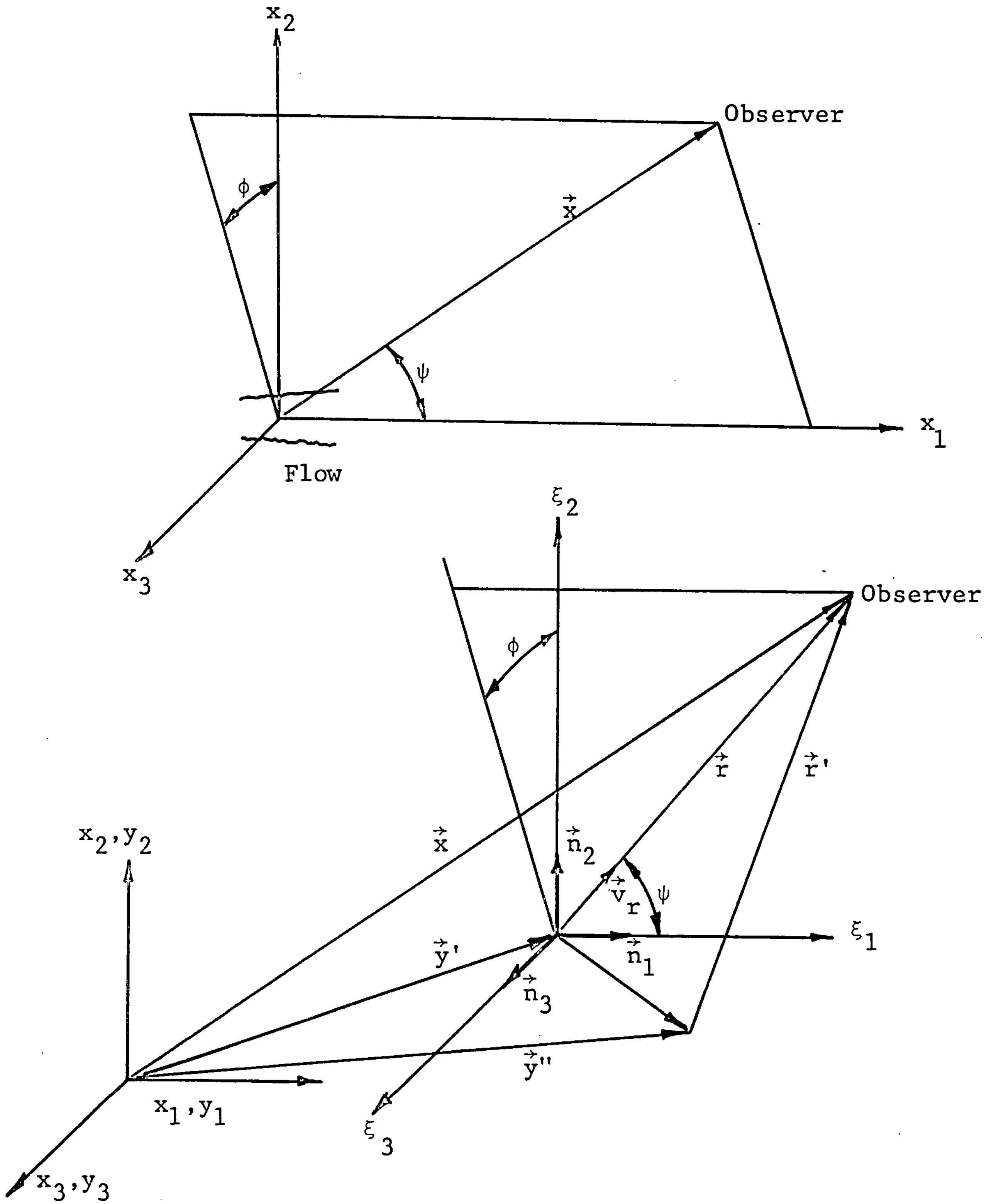

Figure 1. Coordinates of acoustic analysis

and the assumption that the observer distance x is large compared to the flow dimensions, i.e.,

$$c_o \tau \cong \vec{\xi} \cdot \vec{x}/x \tag{3.5}$$

giving

$$I(\psi,\phi) = \int_{\infty} P(\psi,\phi,\vec{y})\, d\vec{y} \tag{3.6}$$

where

$$P(\psi,\phi,\vec{y}) = \frac{\rho_o x_i x_j x_k x_l}{16\pi^2 c_o^5 x^6} \int_{\infty} \frac{\partial^4}{\partial \tau^4} \overline{v_i v_j v_k' v_l'}\, d\vec{\xi} \tag{3.7}$$

Proudman[7] expressed Eq. (3.7) in the very convenient form

$$P(\psi,\phi,\vec{y}) = \frac{\rho_o}{16\pi^2 c_o^5 x^2} \int_{\infty} \frac{\partial^4}{\partial \tau^4} \overline{v_x^2 v_x'^2}\, d\vec{\xi} \tag{3.8}$$

where v_x and v_x' are the components of the velocity fluctuation at $\vec{y}'$ and $\vec{y}''$ in the direction of $\vec{x}$.

The correlation $\overline{v_x^2 v_x'^2}$ can be expressed in terms of the quadrupole correlations $\overline{v_i v_j v_k' v_l'}$ and the coordinates x,ψ,ϕ . Assumption of a normal joint probability distribution for u_i and u_k' permits expressing the fourth-order correlation in the terms of Q_{ij} and Q_{kl} . This will be done in Section 10.

4. DEVELOPMENT OF TWO-POINT, TWO-TIME, TURBULENT CORRELATION EQUATIONS

Here we develop the equations governing the ensemble averaged two-point, two-time correlations of turbulent velocity components that are required to predict aerodynamic sound. As we shall see the equations for the two-point, space-time velocity correlations contain triple-velocity and pressure-velocity correlations which cannot be solved for exactly without recourse to an infinite hierarchy of equations, each containing successively higher-order correlations. This is the so-called closure problem. It will, therefore, be necessary to model these terms in a way which will meet certain criteria. Now, in the limit of zero separation in space and time the exact equations governing the correlations reduce to the equations describing the Reynolds stresses at a point. We shall model the two-point, two-time equations such that at zero separation and time they reduce to an established theory of higher-order closure for the

Reynolds stresses. Such closure theories have been developed by Donaldson[5], Hanjalic and Launder[8], Wolfshtein, Naot and Lin[9] and Lumley and Khajeh-Nouri[10], among others. Closure theories generally specify numerical values of coefficients in their modeled terms chosen to optimize agreement between theory and experimental data. It is considered undesirable to require changes in these coefficients for various flow conditions. The coefficients in our modeled terms, which as we shall see are the viscous, pressure-velocity and triple-velocity correlations, will be assigned those values selected by Donaldson and his associates and employed in their technique of invariant modeling.[5,11] (Henceforth this technique will be referred to as (I) for convenience.)

4.1. One-Point, One-Time Equations

The equations governing the conservation of momentum and mass in an incompressible uniform density fluid with constant viscosity are

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_k^2} \tag{4.1}$$

$$\frac{\partial U_i}{\partial x_i} = 0 \tag{4.2}$$

The velocity U_i is now written as the sum of mean and fluctuating components, $U_i + u_i$, introduced into the governing equations, and an ensemble average is taken. The results are

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = - \frac{\partial \overline{u_i u_j}}{\partial x_j} - \frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j^2} \tag{4.3}$$

$$\frac{\partial U_i}{\partial x_i} = 0 \tag{4.4}$$

These are the equations governing conservation of the mean mass and momentum at one-point and one-time. The equations for the Reynolds stress are developed from Eq. (4.1) and Eq. (4.2) by introducing the mean and fluctuating components for the velocities u_i and u_j and before ensemble averaging, multiplying the respective equations by u_j and u_i. Addition of the equations, subtracting out the mean using Eq. (4.3) and Eq. (4.4) and averaging lead to the equations for the ensemble averaged Reynolds stresses $\overline{u_i u_j}$ at a point.

$$\frac{\partial \overline{u_i u_j}}{\partial t} + U_k \frac{\partial \overline{u_i u_j}}{\partial x_k} = - \overline{u_i u_k} \frac{\partial U_j}{\partial x_k} - \overline{u_j u_k} \frac{\partial U_i}{\partial x_k}$$

$$- \frac{\partial}{\partial x_k} (\overline{u_k u_i u_j}) - \overline{\frac{u_i}{\rho} \frac{\partial p}{\partial x_j}} \tag{4.5}$$

$$- \overline{\frac{u_j}{\rho} \frac{\partial p}{\partial x_i}} + \nu \frac{\partial^2 \overline{u_i u_j}}{\partial x_k^2} - 2\nu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}}$$

These equations are the counterparts of the equations governing the evolution of the two-point, two-time velocity correlations, which will now be derived in more detail. Initially we do not assume homogeneous turbulence, but for simplicity this assumption will be made later, as well as that of a constant gradient mean flow along one of the coordinate axes.

4.2. Two-Point, Two-Time Equations

Now define the mean and instantaneous turbulent fluctuating velocities at point x_k and time t as U_i and u_i, and their counterparts at point x_k' and time t' as U_j' and u_j', introduce them into Eq. (4.1) and subtract out the mean equations. This leaves

$$\frac{\partial u_i}{\partial t} + U_k \frac{\partial u_i}{\partial x_k} + u_k \frac{\partial U_i}{\partial x_k} + u_k \frac{\partial u_i}{\partial x_k} - \frac{\partial}{\partial x_k} \overline{u_i u_k}$$

$$= - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k^2} \tag{4.6}$$

$$\frac{\partial u_j'}{\partial t'} + U_k' \frac{\partial u_j'}{\partial x_k'} + u_k' \frac{\partial U_j'}{\partial x_k'} + u_k' \frac{\partial u_j'}{\partial x_k'} - \frac{\partial}{\partial x_k'} (\overline{u_j' u_k'})$$

$$= - \frac{1}{\rho} \frac{\partial p'}{\partial x_j'} + \nu \frac{\partial^2 u_j'}{\partial x_k'^2} \tag{4.7}$$

Now multiply Eq. (4.6) by u_j' and Eq. (4.7) by u_i. Since primed dependent variables are not functions of unprimed independent variables, and vice versa, they pass through the partial derivatives.

Adding the two equations and taking the ensemble average gives

$$\frac{\partial \overline{u_i u_j'}}{\partial t} + \frac{\partial \overline{u_i u_j'}}{\partial t'} + U_k \frac{\partial \overline{u_i u_j'}}{\partial x_k} + U_k' \frac{\partial \overline{u_i u_j'}}{\partial x_k'} + \overline{u_j' u_k} \frac{\partial U_i}{\partial x_k}$$

$$+ \overline{u_i u_k'} \frac{\partial U_j'}{\partial x_k'} + \frac{\partial \overline{u_i u_k u_j'}}{\partial x_k} + \frac{\partial \overline{u_i u_j' u_k'}}{\partial x_k'} \tag{4.8}$$

$$= - \frac{1}{\rho} \frac{\partial}{\partial x_i} \overline{u_j' p} - \frac{1}{\rho} \frac{\partial}{\partial x_j'} \overline{u_i p'} + \nu \frac{\partial^2 \overline{u_i u_j'}}{\partial x_k^2} + \frac{\partial^2 \overline{u_i u_j'}}{\partial x_k'^2} ,$$

New independent variables are now introduced to differentiate between the effects of absolute position and time and separation distance and time. Define

$$\xi_k = x_k' - x_k \tag{4.9a}$$

$$s_k = \frac{1}{2}[x_k + x_k'] \tag{4.9b}$$

$$\tau = t' - t \tag{4.9c}$$

The derivatives become

$$\frac{\partial}{\partial x_k} = \frac{\partial s_k}{\partial x_k} \frac{\partial}{\partial s_k} + \frac{\partial \xi_k}{\partial x_k} \frac{\partial}{\partial \xi_k} = \frac{1}{2} \frac{\partial}{\partial s_k} - \frac{\partial}{\partial \xi_k} \tag{4.10a}$$

$$\frac{\partial}{\partial x_k'} = \frac{\partial s_k}{\partial x_k'} \frac{\partial}{\partial s_k} + \frac{\partial \xi_k}{\partial x_k'} \frac{\partial}{\partial \xi_k} = \frac{1}{2} \frac{\partial}{\partial s_k} + \frac{\partial}{\partial \xi_k} \tag{4.10b}$$

$$\frac{\partial^2}{\partial x_k^2} = \frac{1}{4} \frac{\partial^2}{\partial s_k^2} + \frac{\partial^2}{\partial \xi_k^2} - \frac{\partial^2}{\partial s_k \partial \xi_k} \tag{4.10c}$$

$$\frac{\partial^2}{\partial x_k'^2} = \frac{1}{4} \frac{\partial^2}{\partial s_k^2} + \frac{\partial^2}{\partial \xi_k^2} + \frac{\partial^2}{\partial s_k \partial \xi_k} \tag{4.10d}$$

$$\frac{\partial}{\partial t'} = \frac{\partial \tau}{\partial t'}\frac{\partial}{\partial \tau} + \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \frac{\partial}{\partial t} \tag{4.10e}$$

Taking the ensemble average we arrive at

$$\begin{aligned}
&2\frac{\partial \overline{u_i u_j'}}{\partial t} + \frac{\partial \overline{u_i u_j'}}{\partial \tau} + \overline{u_j' u_k}\frac{\partial U_i}{\partial x_k} + \overline{u_i u_k'}\frac{\partial U_j'}{\partial x_k'} \\
&\quad + \frac{1}{2}[U_k + U_k']\frac{\partial \overline{u_i u_j'}}{\partial s_k} + [U_k - U_k']\frac{\partial \overline{u_i u_j'}}{\partial \xi_k} \\
&\quad = -\frac{1}{2}\frac{\partial}{\partial s_k}[\overline{u_i u_j' u_k'} + \overline{u_i u_k u_j'}] - \frac{\partial}{\partial \xi_k}[\overline{u_i u_j' u_k'} - \overline{u_i u_k u_j'}] \\
&\quad - \frac{1}{2\rho}\left[\frac{\partial}{\partial s_i}\overline{u_j' p} + \frac{\partial}{\partial s_j}\overline{u_i p'}\right] - \frac{1}{\rho}\left[\frac{\partial}{\partial \xi_j}\overline{u_i p'} - \frac{\partial}{\partial \xi_i}\overline{u_j' p}\right] \\
&\quad + \frac{1}{2}\nu\frac{\partial^2}{\partial s_k^2}\overline{u_i u_j'} + 2\nu\frac{\partial^2}{\partial \xi_k^2}\overline{u_i u_j'}
\end{aligned} \tag{4.11}$$

4.3. Simplifying Assumptions

We now invoke the assumption of homogeneous and stationary turbulence, and all derivatives with respect to the absolute coordinates s_k and absolute time t vanish. Defining $Q_{ij}(\xi_1,\xi_2,\xi_3, ; \tau)$ as $\overline{u_i(x_1,x_2,x_3,t)u_j'(x_1+\xi_1, x_2+\xi_2, x_3+\xi_3 ; t+\tau)}$ we obtain

$$\begin{aligned}
&\frac{\partial Q_{ij}}{\partial \tau} + Q_{jk}\frac{\partial U_i}{\partial x_k} + Q_{ik}\frac{\partial U_j'}{\partial x_k'} + [U_k - U_k']\frac{\partial Q_{ij}}{\partial \xi_k} \\
&\quad = -\frac{\partial}{\partial \xi_k}[\overline{u_i u_j' u_k'} - \overline{u_i u_k u_j'}] - \frac{1}{\rho}\left[\frac{\partial}{\partial \xi_j}\overline{u_i p'} - \frac{\partial}{\partial \xi_i}\overline{u_j' p}\right] \\
&\quad + 2\nu\frac{\partial^2}{\partial \xi_k^2}Q_{ij}
\end{aligned} \tag{4.12}$$

Figure 2 summarizes the coordinates systems in absolute and separation coordinates.

One of our basic working hypotheses is that the behavior of more complex flows may be inferred from the analysis of simpler flows, specifically here that locally the turbulent correlations can be calculated from knowledge of the local velocity gradient. This gradient is taken to be constant over the volume of separation coordinates within which $\overline{u_i u_j}$ becomes completely uncorrelated. The convective velocity in Eq. (4.12) is assumed to be along the ξ_1 coordinate with a constant gradient in the ξ_2 direction. Then

$$U_k = \delta_{1k}\left[U_{1_o} + \xi_2 \frac{dU_1}{dx_2}\right] \tag{4.13}$$

$$U_k - U_k' = \delta_{1k}\xi_2 \frac{dU_1}{dx_2} \tag{4.14}$$

so Eq. (4.12) becomes

$$\begin{aligned}\frac{\partial Q_{ij}}{\partial \tau} + (\delta_{i1}Q_{2j} + \delta_{j1}Q_{i2}) \frac{dU_1}{dx_2} + \xi_2 \frac{dU_1}{dx_2}\frac{\partial Q_{ij}}{\partial \xi_1} \\ = -\frac{\partial}{\partial \xi_k}[\overline{u_i u_j' u_k'} - \overline{u_i u_k u_j'}] + \frac{1}{\rho}\left[\frac{\partial}{\partial \xi_i}\overline{u_j' p} - \frac{\partial}{\partial \xi_j}\overline{u_i p'}\right] \\ + 2\nu \frac{\partial^2}{\partial \xi_k^2} Q_{ij}\end{aligned} \tag{4.15}$$

This equation describes Q_{ij} with respect to a coordinate system ξ_k convected with the local velocity within the shearing flow. If we wish to write the equations for a flow moving at a velocity U_{1_o}, with respect to the coordinate system, then we have

$$\begin{aligned}\frac{\partial Q_{ij}}{\partial \tau} + (\delta_{j1}Q_{2j} + \delta_{j1}Q_{i2}) \frac{dU_1}{dx_2} + \left(U_{1_o} + \xi_2 \frac{dU_1}{dx_2}\right)\frac{\partial Q_{ij}}{\partial \xi_1} \\ = -\frac{\partial}{\partial \xi_k}[\overline{u_i u_j' u_k'} - \overline{u_i u_k u_j'}] + \frac{1}{\rho}\left[\frac{\partial}{\partial \xi_i}\overline{u_j' p} - \frac{\partial}{\partial \xi_j}\overline{u_i p'}\right] \\ + 2\nu \frac{\partial^2}{\partial \xi_k^2} Q_{ij}\end{aligned} \tag{4.16}$$

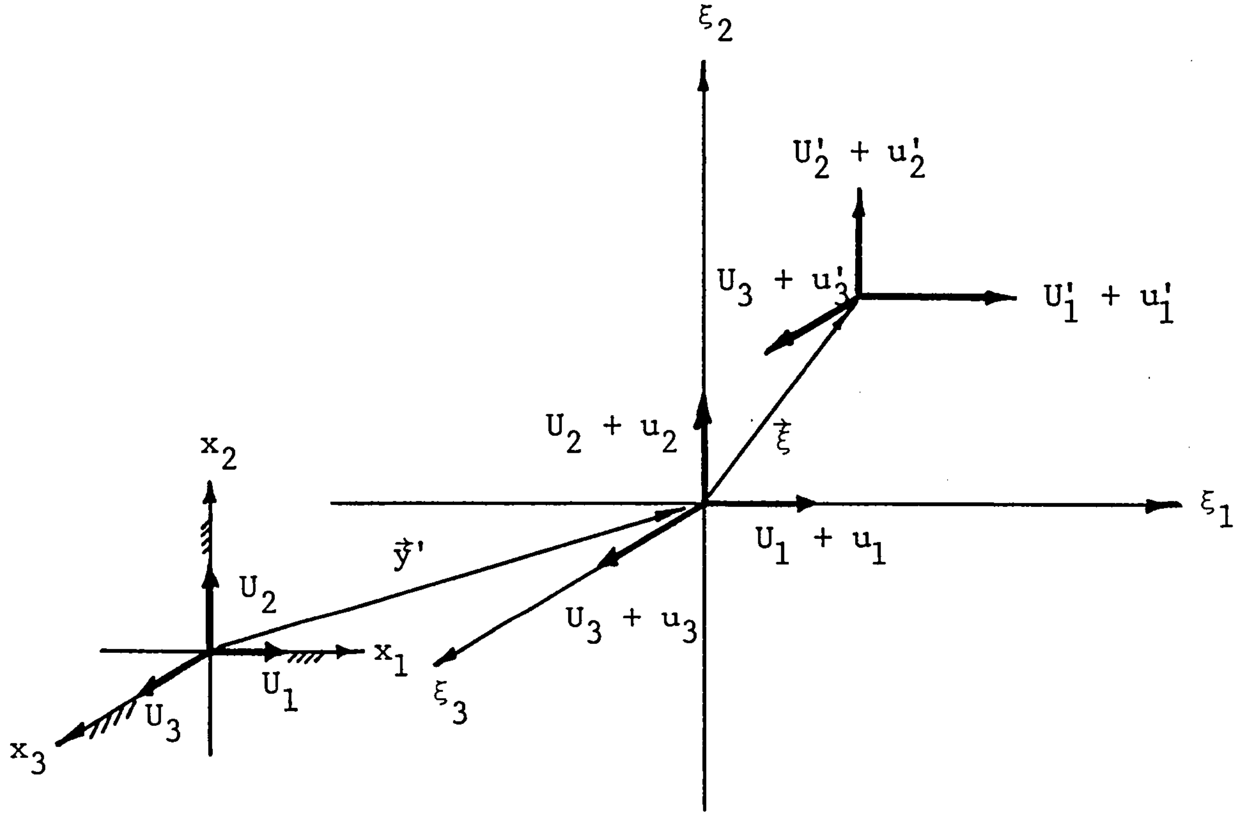

Figure 2. Absolute and separation coordinates for two-point, two-time turbulent velocity correlations

Now, for homogeneous turbulence in the coordinate system adopted and since the turbulence is stationary, the correlations will not be changed if we replace ξ_k by $-\xi_k$ and τ by $-\tau$. From the condition of invariance under translation, for any homogeneous flow field*

$$\overline{u_i u_j'}\ (\xi_1,\xi_2,\xi_3;\tau) = \overline{u_j u_i'}\ (-\xi_1,-\xi_2,-\xi_3;-\tau)$$

$$\overline{u_i u_k' u_j'}\ (\xi_1,\xi_2,\xi_3;\tau) = \overline{u_i' u_k u_j}\ (-\xi_1,-\xi_2,-\xi_3;-\tau)$$

$$\overline{u_i u_k u_j'}\ (\xi_1,\xi_2,\xi_3;\tau) = \overline{u_i' u_k' u_j}\ (-\xi_1,-\xi_2,-\xi_3;-\tau)$$

$$\overline{p u_j'}\ (\xi_1,\xi_2,\xi_3;\tau) = \overline{p' u_j}\ (-\xi_1,-\xi_2,-\xi_3;-\tau)$$

From the condition of invariance under reflection for this coordinate system

$$\overline{u_i u_j'}\ (\xi_1,\xi_2,\xi_3;\tau) = \overline{u_i u_j'}\ (-\xi_1,-\xi_2,-\xi_3;-\tau)$$

$$\overline{u_i u_k' u_j'}\ (\xi_1,\xi_2,\xi_3;\tau) = -\ \overline{u_i u_k' u_j'}\ (-\xi_1,-\xi_2,-\xi_3;-\tau)$$

$$\overline{u_i u_k u_j'}\ (\xi_1,\xi_2,\xi_3;\tau) = -\ \overline{u_i u_k u_j'}\ (-\xi_1,-\xi_2,-\xi_3;-\tau)$$

$$\overline{p u_j'}\ (\xi_1,\xi_2,\xi_3;\tau) = -\ \overline{p u_j'}\ (-\xi_1,-\xi_2,-\xi_3;-\tau)$$

The triple-velocity correlations are symmetric with respect to the indices referring to the same point so that

$$\overline{u_i u_k u_j'} = \overline{u_k u_i u_j'}$$

$$\overline{u_i u_k' u_j'} = \overline{u_i u_j' u_k'}$$

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*The following symmetry arguments can be found for $\tau = 0$ in Hinze,[12] p. 332.

The term in Eq. (4.17) containing the triple-velocity correlation term gives

$$\frac{\partial}{\partial \xi_k} [\overline{u_i u_j' u_k'} - \overline{u_i u_k u_j'}] \Bigg|_{\substack{\xi_k = 0 \\ \tau = 0}} = 0$$

From continuity, contraction of the pressure-velocity correlation term gives

$$\frac{\partial}{\partial \xi_k} \overline{u_i p'} - \frac{\partial}{\partial \xi_i} \overline{u_i' p} = 0$$

Contraction of Eq. (4.16) then yields

$$\frac{\partial Q_{ii}}{\partial \tau} + 2Q_{12} \frac{dU_1}{dx_2} + \left(U_{1_o} + \xi_2 \frac{dU_1}{dx_2}\right) \frac{\partial Q_{ii}}{\partial \xi_1}$$

$$= - \frac{\partial}{\partial \xi_k} [\overline{u_i u_i' u_k'} - \overline{u_i u_k u_i'}] + 2\nu \frac{\partial^2 Q_{ii}}{\partial \xi_k^2} \tag{4.17}$$

Now at zero separation $\partial^2 Q_{ii} / \partial \xi^2$ can be written

$$\frac{\partial^2}{\partial \xi_k^2} Q_{ii} \Bigg|_o = - \frac{\partial^2}{\partial x_k \partial x_k'} \overline{u_i u_i'} = - \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_i'}{\partial x_k'}}$$

so that for $\xi_k = 0, \tau = 0$, Eq. (4.17) reduces to

$$\frac{\partial \overline{u_i u_i}}{\partial \tau} + 2\overline{u_1 u_2} \frac{dU_1}{dx_2} + U_{1_o} \frac{\partial \overline{u_i u_i}}{\partial \xi_1} = - 2\nu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_k}}$$

Since $\overline{u_i u_i}(\xi_k ; \tau) = \overline{u_i u_i}(-\xi_k ; -\tau)$

$$\overline{u_1 u_2} \frac{dU_1}{dx_2} = - \nu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_k}} \tag{4.18}$$

at $\xi_k = 0$, $\tau = 0$.

To facilitate writing equations let

$$P_{ij} = \left[\frac{\partial}{\partial \xi_j} \overline{u_i p'} - \frac{\partial}{\partial \xi_i} \overline{u_j' p}\right] \tag{4.19}$$

$$S_{ij} = \frac{\partial}{\partial \xi_k} [\overline{u_i u_j' u_k'} - \overline{u_i u_k u_j'}] \tag{4.20}$$

Written out in full Eqs. (4.16) now appear as

$$\frac{\partial Q_{11}}{\partial \tau} + \left(U_{1_o} + \xi_2 \frac{dU_1}{dx_2}\right) \frac{\partial Q_{11}}{\partial \xi_1} = - 2Q_{12} \frac{dU_1}{dx_2} - S_{11} - \frac{1}{\rho} P_{11} + 2\nu \nabla^2 Q_{11} \tag{4.21a}$$

$$\frac{\partial Q_{22}}{\partial \tau} + \left(U_{1_o} + \xi_2 \frac{dU_1}{dx_2}\right) \frac{\partial Q_{22}}{\partial \xi_1} = - S_{22} - \frac{1}{\rho} P_{22} + 2\nu \nabla^2 Q_{22} \tag{4.21b}$$

$$\frac{\partial Q_{33}}{\partial \tau} + \left(U_{1_o} + \xi_2 \frac{dU_1}{dx_1}\right) \frac{\partial Q_{33}}{\partial \xi_1} = - S_{33} - \frac{1}{\rho} P_{33} + 2\nu \nabla^2 Q_{33} \tag{4.21c}$$

$$\frac{\partial Q_{12}}{\partial \tau} + \left(U_{1_o} + \xi_2 \frac{dU_1}{dx_2}\right) \frac{\partial Q_{12}}{\partial \xi_1} = - Q_{22} \frac{dU_1}{dx_2} - S_{12} - \frac{1}{\rho} P_{12} + 2\nu \nabla^2 Q_{12} \tag{4.21d}$$

Although equations may be written for the Q_{13} and Q_{23} correlations (which are equal to zero at zero separation), they may be found from continuity, given the remaining correlations. We shall neglect Q_{13} and Q_{23} in this initial study for simplicity, concentrating only on the correlations which are explicitly required to compute the energy components Q_{11} , Q_{22} and Q_{33} .

Although it is not possible to define precisely the characteristic physical behavior of each term in the governing equations for the two-point, two-time velocity correlations, they may be identified by analogy with the one-point, one-time equations as follows. Write the equations as

$$\frac{\partial Q_{ij}}{\partial \tau} + \left(U_{1_o} + \xi_2 \frac{dU_1}{dx_2}\right) \frac{\partial Q_{ij}}{\partial \xi_1} = - \underbrace{\left(\delta_{i1} Q_{2j} + \delta_{j1} Q_{i2}\right)}_{\mathrm{I}} \frac{dU_1}{dx_2}$$
$$- \frac{1}{\rho} \underbrace{\left[\frac{\partial}{\partial \xi_j} \overline{u_i p'} - \frac{\partial}{\partial \xi_i} \overline{u_j' p}\right]}_{\mathrm{II}} - \frac{\partial}{\partial \xi_k} \underbrace{[\overline{u_i u_j u_k'} - \overline{u_i u_k u_j'}]}_{\mathrm{III}}$$
$$+ \underbrace{2\nu \frac{\partial^2}{\partial \xi_k^2} Q_{ij}}_{\mathrm{IV}} \tag{4.22}$$

I is a production-like term, where interaction with the mean flow generates a net increase in the u_1u_1' and u_1u_2' correlations. II, the pressure-velocity correlations, disappear upon contraction. This term transfers correlations between components. In one-point, one-time theory it is the "tendency-towards-isotropy" term. Since production of energy-containing eddies occurs in the $i=j=1$ equation, this term transfers energy to the 22 and 33 components which in turn may exchange energy among themselves and the 12 correlations. Deissler[13] and Fox[14] have studied this transfer for low Reynolds number turbulence. A summary of their results appears in Hinze[12].

III is the triple-velocity correlation and is a diffusion-like transfer of Q_{ij} by turbulence gradients. Finally, IV is a diffusion-like term which serves to decorrelate Q_{ij} through the action of viscous stresses. At zero separation it reduces to the dissipation, $-2\nu \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}$, and the triple-velocity correlation term is zero. Thus, the contracted form of the equation, for which the pressure-velocity terms sum to zero, shows that the dissipation must be balanced by the production for stationary turbulence.

5. MODELING OF PRESSURE-VELOCITY, TRIPLE-VELOCITY AND DISSIPATION CORRELATIONS AT SECOND-ORDER

In this initial study, intended to test the overall validity of our approach, it is desirable to provide closure of the governing equations using the simplest possible model consistent with physical principles. The philosophy here is to extend the form of one-point, one-time modeling to the two-point, two-time correlations such that the latter reduce to the former for $\xi_k = 0$, $\tau = 0$.

5.1. Modeling of Triple-Velocity Correlations

The triple-velocity correlations at zero separation in space and time have usually been modeled as gradient diffusion terms in previous closure techniques. The form adopted here and which has been used in (I) is

$$\frac{\partial}{\partial x_k} (\overline{u_i u_j u_k}) = - \frac{\partial}{\partial x_k} \left(v_c q \Lambda \frac{\partial \overline{u_i u_j}}{\partial x_k} \right) \qquad (5.1)$$

where $q^2 = \overline{u_i u_i}$ and Λ is a turbulent length scale. In anisotropic turbulence we should expect that Λ is dependent upon direction. Indeed, this would be the natural extention of Eq. (5.1) to a more sophisticated model. Later, we shall define a set of directionally dependent turbulent scales. In the present turbulent diffusion model Λ will be assumed independent of direction, although we might expect Λ to be dominated by the assumed unidirectional mean velocity gradient in the x_2 direction.

Equation (5.1) satisfies the tensor symmetry of the triple-velocity derivatives $\partial/\partial x_k(\overline{u_i u_j u_k})$ but not the symmetry of $u_i u_j u_k$. The model coefficient v_c was assigned the value of 0.3 in (I). This model is extended to a two-point, two-time modeling by writing $\overline{u_i u_j}$ in place of the one-point, one-time correlation. Due to homogeneity the turbulent diffusion coefficient $v_c q\Lambda$ is assumed uniform in space and so

$$S_{ij} = - v_c q\Lambda\nabla^2 Q_{ij} \tag{5.2}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial\xi_1^2} + \frac{\partial^2}{\partial\xi_2^2} + \frac{\partial^2}{\partial\xi_3^2}$$

5.2. Modeling of Pressure-Velocity Correlations

As noted previously, contraction of the governing equations results in elimination of the pressure-velocity correlations since $P_{ii} = 0$ in homogeneous turbulence. For one-point, one-time correlations these terms redistribute energy between the velocity components, tending to decrease anisotropy and decrease the turbulent shear stress. According to Deissler and Fox, P_{ij} may actually promote anisotropy in two-point correlations under some circumstances.

From the governing equations for $i = j$ we see that the production term appears only in the Q_{11} equation. P_{ij} must therefore increase Q_{22} and Q_{33} at the expense of Q_{11}. Rotta[15] modeled the pressure-velocity correlations as a tendency-towards-isotropy term for one-point, one-time calculations as

$$\frac{1}{\rho}\left[\overline{u_i\frac{\partial p}{\partial x_j}} + \overline{u_j\frac{\partial p}{\partial x_i}}\right] = C\frac{q}{\Lambda}\left[\overline{u_i u_j} - \frac{1}{3}\delta_{ij}q^2\right] \tag{5.3}$$

where C is an order one constant. Taking $C = 1$, the form of Eq. (5.3) is extended for the present analysis to

$$\frac{1}{\rho}P_{ij} = \frac{q}{\Lambda}\left[Q_{ij} - \frac{1}{3}\delta_{ij}Q_{\ell\ell}\right] \tag{5.4}$$

which reduces to Rotta's form at zero separation. Note that Q_{ij} could be transferred unequally among components by writing Eq. (5.4) as

$$\frac{1}{\rho}P_{ij} = \frac{q}{\Lambda}\left[Q_{ij} - \eta_{ij}Q_{\ell\ell}\right] \tag{5.4a}$$

where we must set $\eta_{ii} = 1$ to make $P_{ii} = 0$. For simplicity in this initial study we shall use Eq. (5.4).

5.3. Modeling of the Dissipation Term

Term IV of Eq. (4.22), $+2\nu \partial^2 Q_{ij}/\partial\xi_k^2$, is a diffusion-like transfer of correlations due to viscous effects. At one-point, and one-time this terms becomes $-2\nu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}}$, and under contraction of the indices balances total production. For large Reynolds number the magnitude of this dissipation forces a high rate of decorrelation in Q_{ij} near zero separation, and therefore the second derivatives of Q_{ij} are large in this region. If we express Q_{ij} near zero separation as

$$Q_{ij} = (\overline{u_i u_j})_o \left[1 - \frac{\xi_i \xi_i}{\lambda^2} + \cdots\right] \tag{5.5}$$

where λ is the turbulent microscale, then the dissipation is proportional to $-(\overline{u_i u_j})_o/\lambda^2$. As suggested by Rotta[15,16], the microscale is modeled as

$$\lambda^2 = \frac{\Lambda^2}{a + \frac{bq\Lambda}{\nu}} \tag{5.6}$$

Therefore, in the high Reynolds number limit the dissipation term will be

$$2\nu \frac{\partial^2 Q_{ij}}{\partial \xi_k^2} = - 2b\frac{q}{\Lambda} Q_{ij} \tag{5.7}$$

where $(\overline{u_i u_j})$ has been generalized to Q_{ij}. The one-point, one-time limit of the contracted form of this model is $-2bq^3/\Lambda$. Donaldson in (I) has used the isotropic form of this model, (i.e., $2/3\delta_{ij} bq^3/\Lambda$), with $b = 0.125$. For non-zero spatial separation we might expect a smaller value of b since the viscous decorrelation occurs over a larger characteristic length than λ. For increasing separation in time it is reasonable to suppose that decorrelation will increase. For this study we shall assume that b is some average value over separation space and a function of τ. A preliminary value will be assigned at some point as well as coefficients in the separation-time expansion. These values will be chosen to provide physically realizable behavior of the correlations and to match experimental acoustic measurements.

5.4. Modeled Form of Equations

Equations (4.22) now take the form

$$\frac{\partial Q_{ij}}{\partial \tau} + \left(U_{1_o} + \xi_2 \frac{dU_1}{dx_2}\right)\frac{\partial Q_{ij}}{\partial \xi_1} = -\left(\delta_{i1}\, Q_{2j} + \delta_{j1} Q_{i2}\right)\frac{dU_1}{dx_2}$$

$$- \frac{q}{\Lambda}\left[Q_{ij} - \frac{1}{3}\delta_{ij}Q_{\ell\ell}\right] + v_c q\Lambda\nabla^2 Q_{ij} - \frac{2bq}{\Lambda}\, Q_{ij} \qquad (5.8)$$

Written out in full these are

$$\frac{\partial Q_{11}}{\partial \tau} + \left(U_{1_o} + \xi_2 \frac{dU_1}{dx_2}\right)\frac{\partial Q_{11}}{\partial \xi_1} = -\, 2Q_{12}\,\frac{dU_1}{dx_2} - \frac{q}{3\Lambda}\left[2Q_{11} - Q_{22} - Q_{33}\right]$$

$$+ v_c q\Lambda\nabla^2 Q_{11} - \frac{2bq}{\Lambda}\, Q_{11} \qquad (5.9a)$$

$$\frac{\partial Q_{22}}{\partial \tau} + \left(U_{1_o} + \xi_2 \frac{dU_1}{dx_2}\right)\frac{\partial Q_{22}}{\partial \xi_1} = -\, \frac{q}{3\Lambda}\left[2Q_{22} - Q_{11} - Q_{33}\right]$$

$$+ v_c q\Lambda\nabla^2 Q_{22} - \frac{2bq}{\Lambda}\, Q_{22} \qquad (5.9b)$$

$$\frac{\partial Q_{33}}{\partial \tau} + \left(U_{1_o} + \xi_2 \frac{dU_1}{dx_2}\right)\frac{\partial Q_{33}}{\partial \xi_1} = -\, \frac{q}{3\Lambda}\left[2Q_{33} - Q_{11} - Q_{22}\right]$$

$$+ v_c q\Lambda\nabla^2 Q_{33} - \frac{2bq}{\Lambda}\, Q_{33} \qquad (5.9c)$$

$$\frac{\partial Q_{12}}{\partial \tau} + \left(U_{1_o} + \xi_2 \frac{dU_1}{dx_2}\right)\frac{\partial Q_{12}}{\partial \xi_1} = -\, Q_{22}\,\frac{dU_1}{dx_2} - \frac{q}{\Lambda}\, Q_{12}$$

$$+ v_c q\Lambda\nabla^2 Q_{12} - \frac{2bq}{\Lambda}\, Q_{12} \qquad (5.9d)$$

6. ONE-POINT, ONE-TIME TURBULENT VELOCITY CORRELATIONS FOR STATIONARY, HOMOGENEOUS SIMPLE SHEAR FLOW

In applying the present theory to calculate turbulent acoustics the values of q and Λ in Eqs. (5.9) would be obtained from full numerical solutions of the flow being studied. For convenience, and to provide a consistent set of one-point, one-time velocity correlations that will be required in order to evaluate the shear flow model we are developing here, let us examine the limiting form of the governing equations for $(\overline{u_i u_j})$ at $\xi_k = 0$, $\tau = 0$. These correlations are the initial conditions for the two-point, two-time correlations. What follows is adapted from Donaldson's superequilibrium theory in (I).

Since the flow is homogeneous and stationary, derivatives of $(\overline{u_i u_j})_o$ with respect to absolute space and time are zero. At the limit of zero separation in space and time, Eqs. (4.5) define the one-point, one-time turbulent velocity correlations for this flow. The left hand sides of the equations vanish. The viscous term transforms to a dissipation term at zero separation, dependent upon the curvature in separation space, of the two-point velocity correlations at zero separation. The equations at zero separation become

$$0 = -(\delta_{i1}\overline{u_2 u_j} + \delta_{j1}\overline{u_i u_2})\frac{dU_1}{dx_2} - \frac{q}{\Lambda}[\overline{u_i u_j} - \tfrac{1}{3}\delta_{ij}q^2] - 2\nu\,\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}} \tag{6.1}$$

where the modeled form of the pressure-velocity correlations is used.

The contracted form of this equation yields

$$\overline{u_1 u_2}\frac{dU_1}{dx_2} = -\nu\,\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k}} = \varepsilon \tag{6.2}$$

or production exactly balances dissipation, where ε is the total dissipation rate of the turbulent kinetic energy per unit mass, $1/2q^2$. If we define the dissipation terms in general as

$$\varepsilon_{ij} = \nu\,\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}} \tag{6.3}$$

then the equations may be written

$$2\varepsilon - \frac{q}{3\Lambda}\,[2\overline{u_1u_1} - \overline{u_2u_2} - \overline{u_3u_3}] - 2\varepsilon_{11} = 0 \qquad (6.4a)$$

$$- \frac{q}{3\Lambda}\,[2\overline{u_2u_2} - \overline{u_1u_1} - \overline{u_3u_3}] - 2\varepsilon_{22} = 0 \qquad (6.4b)$$

$$- \frac{q}{3\Lambda}\,[2\overline{u_3u_3} - \overline{u_1u_1} - \overline{u_2u_2}] - 2\varepsilon_{33} = 0 \qquad (6.4c)$$

$$- \overline{u_2u_2}\,\frac{dU_1}{dx_2} - \frac{q}{\Lambda}\,\overline{u_1u_2} \qquad - 2\varepsilon_{12} = 0 \qquad (6.4d)$$

and the contracted equations show, of course, that

$$\varepsilon = \varepsilon_{ii} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \qquad (6.5)$$

Equations (6.4a), (6.4b) and (6.4c) are, therefore, not independent since any one can be derived from the other two.

Manipulation of Eqs. (6.4) yields the velocity correlations in terms of the dissipation

$$\frac{\overline{u_1u_1}}{q^2} = \frac{1}{3} + \frac{2\Lambda}{q^3}\,(\varepsilon_{22} + \varepsilon_{33}) \qquad (6.6a)$$

$$\frac{\overline{u_2u_2}}{q^2} = \frac{1}{3} - \frac{2\Lambda}{q^3}\,\varepsilon_{22} \qquad (6.6b)$$

$$\frac{\overline{u_3u_3}}{q^2} = \frac{1}{3} - \frac{2\Lambda}{q^3}\,\varepsilon_{33} \qquad (6.6c)$$

$$\frac{\overline{u_1u_2}}{q^2} = - \frac{\Lambda\,\frac{dU_1}{dx_2}}{q}\left[\frac{1}{3} - \frac{2\Lambda}{q^3}\,\varepsilon_{22}\right] - 2\,\frac{\Lambda}{q^3}\,\varepsilon_{12} \qquad (6.6d)$$

We can solve for ε_{12} in terms of the other dissipation components. First let us define

$$A = \frac{q}{\Lambda \frac{dU_1}{dx_2}} \tag{6.7}$$

which for a constant value of A is a definition of Λ. The ratio $q/dU_1/dx_2)$ can be considered a characteristic length in a constant shear turbulent flow. The inverse of A then defines the size of a "typical" eddy in terms of this characteristic length.

ε_{12} then becomes

$$\varepsilon_{12} = \frac{1}{A}\frac{q^3}{\Lambda}\left[(\frac{1}{2} A^2 \varepsilon + \varepsilon_{22}) \frac{\Lambda}{q^3} - \frac{1}{6} \right] \tag{6.8}$$

In shear flow we would expect ε_{ij} to be anisotropic. Past investigation of shear flow have usually assumed isotropic dissipation

$$\varepsilon_{ij} = \frac{1}{3}\delta_{ij} b \frac{q^3}{\Lambda} \tag{6.9}$$

For simplicity we shall also use this assumption and Eqs. (6.6) give

$$\frac{\overline{u_1 u_1}}{q^2} = \frac{1}{3}[1 + 4b] \tag{6.10a}$$

$$\frac{\overline{u_2 u_2}}{q^2} = \frac{\overline{u_3 u_3}}{q^2} = \frac{1}{3}[1 - 2b] \tag{6.10b,c}$$

$$\frac{\overline{u_1 u_2}}{q^2} = - \frac{1}{3A}[1 - 2b] \tag{6.10d}$$

For $\varepsilon_{12} = 0$, Eq. (6.8) gives A in terms of b

$$A = \left[\frac{1 - 2b}{3b}\right]^{\frac{1}{2}} \tag{6.11}$$

In (I), b was assigned a value of 0.125, and therefore $A = \sqrt{2}$. Table 1 presents a comparison between the results of experimental investigations of nearly homogeneous, constant shear flow turbulence by Champagne, Harris and Corrsin[17], and Harris, Graham and Corrsin[18] and the nondimensional turbulent velocity correlations predicted by Eqs. (6.10) and Eq. (6.11) for $b = 0.125$.

Table 1

Comparison Between Measured and Theoretical Turbulent Velocity Correlations for Constant Mean Shear Flow

Turbulent Velocity Correlation	DATA Champagne, et al.[17]	Harris et al.[18]	THEORY (b = 0.125)
$\overline{u_1u_1}/q^2$	0.47	0.50	0.50
$\overline{u_2u_2}/q^2$	0.25	0.20	0.25
$\overline{u_3u_3}/q^2$	0.28	0.30	0.25
$\overline{u_1u_2}/q^2$	-0.16	-0.15	-0.176

Agreement between theory and data is acceptable for our present purposes.

Having an approximation for the zero-separation limit we now can go on to select the functional representation for the two-point, two-time correlations.

7. SELECTION OF TWO-POINT, SPACE TIME CORRELATION FUNCTION

One of the basic requirements of our analysis is that we can select a functional form for $Q_{ij}(\xi_1,\xi_2,\xi_3;\tau)$ that is both versatile enough to characterize the behavior of the two-point, two-time velocity correlations, yet be amenable to analysis. The trial function that we select should be functionally explicit in separation space (so that spatial integration can be accomplished analytically) and contain a set of separation-time-dependent parameters. The form of the selected function is based on examination of experimental measurements of two-point, two-time velocity correlations, supplemented by theoretical conditions which may be inferred from continuity, symmetry, homogeneity and stationarity of the turbulence and the limiting form of the correlations at zero and infinite separation in space and time.

The method of solution then consists of forming a set of equations based on taking the (m,k)-fold spatial moments of the governing equations containing the selected trial function, where m refers to the moments in the spatial coordinates and k the directional dependence and then integrating in separation space to form equations dependent only on τ . The product $m \times k$ must be chosen such that the number of equations formed, in conjunction with conditions derived from supplementary constraints such as continuity, are sufficient to calculate the total number of parameters contained in the trial function.

Since in any specific turbulent flow the characteristic size of the energy-containing eddies is set physically, such as by the largest dimension of the flow, by the diameter of a jet, etc., we must leave one parameter of the analysis free to be specified. In general, this should be one of the integral scales, and this is the approach taken here.

7.1. Supplementary Constraints

The form of the two-point, two-time velocity correlation trial function must conform to several conditions

1. Loss of correlation at large separation in space and time

$$Q_{ij}(\xi_k \to \infty \; ; \; \tau) \to 0 \tag{7.1a}$$

$$Q_{ij}(\xi_k \; ; \; \tau \to \infty) \to 0 \tag{7.1b}$$

2. One-point, one-time limit

$$Q_{ij}(0 \; ; \; 0) = (\overline{u_i u_j})_o \tag{7.2}$$

3. Incompressibility

$$\frac{\partial Q_{ij}}{\partial \xi_i} = \frac{\partial Q_{ij}}{\partial \xi_j} = 0 \tag{7.3a,b}$$

4. Symmetry with respect to the ξ_1 , ξ_2 plane for unidirectional shear flow

$$\left. \frac{\partial^n Q_{ij}}{\partial \xi_3^n} \right|_{\xi_3=0} = 0 \qquad \text{(n odd)} \tag{7.4}$$

5. Homogeneity and stationarity of turbulence. In coordinates convected with the mean flow

$$Q_{ij}(\xi_k ; \tau) = Q_{ij}(-\xi_k ; -\tau) \tag{7.5}$$

6. Taylor's hypothesis; temporal history at a fixed point is related to the convected spatial structure by

$$\left(\frac{\partial \overline{u_i u_j}}{\partial \tau}\right)^2 \cong U_{1_o}^{\ 2} \left(\frac{\partial \overline{u_i u_j}}{\partial \xi_1}\right)^2 \tag{7.6}$$

The decay of Q_{ij} with τ in convected coordinates makes this only approximate. A memory function $g_{ij}(\tau)$ will be used to account for this decay in separation time in a convected coordinate system.

7.2. Previous Approximate Functions

Frenkiel[19,20] evaluated a family of approximations for the isotropic lateral velocity correlation $g(\xi)$ of the form

$$g(\xi) = \psi(\xi) \exp \left[-|c\xi|^m\right] \tag{7.7}$$

where

$$\psi(\xi) = a_o + \sum a_n \cos (m_n c\xi) \tag{7.8a}$$

or

$$\psi(\xi) = 1 + \sum a_n c^n |\xi|^m \tag{7.8b}$$

He found adequate agreement with wind tunnel data could be achieved near $\xi = 0$ only for $m > 2$, but over the whole curve an adequate approximation could be found for $1 < m < 2$.

For aerodynamic sound analyses, Ribner[21,22] approximated the fluctuating pressure correlation using the Gaussian form

$$\overline{p^2} = \overline{p_o^2} \exp \left[-a_1^2(\xi_1 - U\tau)^2 - a_2^2\xi_2^2 - a_3^2\xi_3^2 - \alpha^2 a_1^2 U^2 \tau^2\right] \tag{7.9}$$

where provision for different turbulent scales is made in the factors a_1 , a_2 and a_3 .

Several investigators have based their calculations on locally homogeneous isotropic turbulence correlation models.[1,22,23,24,25] Goldstein and Rosenbaum[26] treated a more refined model of axisymmetric turbulence, using a zero time delay function similar to Eq. (7.7) with $m = 1$ and $\phi(x) = 1$.

7.3. Present Model

The trial function selected here combines some of the features of the Frenkiel and Ribner models described above. The rationale underlying its adoption is discussed below and its limitations will be presented shortly.

The model we will use here is

$$Q_{ij}(\xi_1,\xi_2,\xi_3;\tau) = (\overline{u_i u_j})_o\, R_{ij}(\xi_1,\xi_2,\xi_3;\tau)\; g_{ij}(\tau) \qquad (7.10a)$$

$$R_{ij}(\xi_1,\xi_2,\xi_3;\tau) = \Big[1 - \alpha_{ij}\left(\xi_1 - U_{1_o}\tau\right)^2 - \beta_{ij}\xi_2^2 - \gamma_{ij}\xi_3^2 - \mu_{ij}\left(\xi_1 - U_{1_o}\tau\right)\xi_2\Big]\cdot\exp\left[-\frac{\left(\xi_1 - U_{1_o}\tau\right)^2}{\sigma_{ij1}^2} - \frac{\xi_2^2}{\sigma_{ij2}^2} - \frac{\xi_3^2}{\sigma_{ij3}^2}\right] \qquad (7.10b)$$

where there is no summation implied in Eq. (7.10a). The spread σ_{ijk} for the ij correlation in the k^{th} coordinate direction is simplified somewhat by writing it

$$\sigma_{ijk} = \sigma_{ij}c_k \qquad (7.11)$$

Thus, the anisotropy in spread is assumed equal among the different correlations.

The model guarantees that the correlations approach zero at large space separation since they decrease at least exponentially with ξ_k^2 . $g_{ij}(\tau)$, the memory function, may be calculated numerically but will be evaluated near $\tau = 0$ as we show later. Q_{ij} approaches zero exponentially with τ^2 and with $g_{ij}(\tau)$ unless we are translating with the flow at velocity U_{1_o} , in which case loss of correlation is dependent primarily on $g_{ij}(\tau)$. The gradient of $U(\tau_2)$ is neglected in the trial function for this initial study. It will be added later if it appears that doing so can improve the present model.

The bracketed polynomial in Eq. (7.10) permits zero crossings which are necessary to satisfy continuity. Equation (7.10) does not do this in the differential sense (Eq. (7.2a or b)) but we force continuity in an integral sense as follows. From Eq. (7.3a) we integrate the equation over half-space for each coordinate direction

$$\int_{-\infty}^{\infty} d\xi_2 \int_{-\infty}^{\infty} d\xi_3 \int_{-\infty}^{0} \frac{\partial Q_{\ell m}}{\partial \xi_\ell} \, d\xi_1 = 0 \tag{7.12a}$$

$$\int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{\infty} d\xi_3 \int_{-\infty}^{0} \frac{\partial Q_{\ell m}}{\partial \xi_\ell} \, d\xi_2 = 0 \tag{7.12b}$$

$$\int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{\infty} d\xi_2 \int_{-\infty}^{0} \frac{\partial Q_{\ell m}}{\partial \xi_\ell} \, d\xi_3 = 0 \tag{7.12c}$$

The alternate statement of continuity, Eqs. (7.3b), is not independent of Eq. (7.a) (see Batchelor[27], p. 27). The results of Eqs. (7.12) are five independent equations in the model parameters.

The continuity constraints require that no net flow passes through the three perpendicular coordinate planes passing through $\xi = 0$. Since we may translate these planes freely in the homogeneous flowfield without effect, these conditions are met everywhere. Note that these conditions are τ dependent.

Q_{ij} is expressed in Eq. (7.10a) as the product of the zero separation, one-point, one-time double-velocity correlation $(u_i u_j)_o$, a normalized function which we designate as $R_{ij}(\xi_1, \xi_2, \xi_3, \tau)$, and the normalized memory function $g_{ij}(\tau)$. $R_{ij}(0,0,0;0) = 1$, and we specify $g_{ij}(0) = 1$.

Taylor's hypothesis is satisfied by writing the flow direction argument as $(\xi_1 - U_{1o}\tau)^2$. R_{ij} is symmetric in this argument since τ changes sign with ξ_1 . In calculating the dependent parameters it is important to separate the variations that occur due to convective effects, i.e., separation coordinate changes, from variations that occur in convected coordinates. It is the latter that constitute the actual variations that are of interest. Thus, in solving for the time variation of the function parameters the convective velocity is set equal to zero. The decrease in Q_{ij} with convective separation is accounted for by the convective argument $(\xi_1 - U_{1o}\tau)$. $g_{ij}(\tau)$ provides decorrelation in the coordinate system translating with the flow.

One of the c_k functions is set equal to one both initially and for $\tau > 0$. At $\tau = 0$ we specify one length scale in the same direction. For $\tau > 0$ the memory function is used to provide the remaining time dependence of the length scale in the appropriate direction. The σ_{ij} spread functions are not taken to be time dependent, being set by specification of the initial conditions. These initial conditions are calculated from the governing equations with separation time derivative equal to zero. ($Q_{ij}/\partial\tau = 0$ at $\tau = 0$ since the flow is assumed homogeneous and stationary). Note that initially only one parameter is specified, the integral scale of turbulence for one ij correlation in one coordinate direction. The integral scales in the $k = 1, 2$ and 3 directions are defined by (no summation)

$$\Lambda_{ij}^{(1)}(\tau) = g_{ij}(\tau)\int_0^\infty R_{ij}(\xi_1 - U\tau,0,0;\tau)d(\xi_1 - U\tau) \tag{7.13a}$$

$$\Lambda_{ij}^{(2)}(\tau) = g_{ij}(\tau)\int_0^\infty R_{ij}(0,\xi_2,0;\tau)d\xi_2 \tag{7.13b}$$

$$\Lambda_{ij}^{(3)}(\tau) = g_{ij}(\tau)\int_0^\infty R_{ij}(0,0,\xi_3;\tau)d\xi_3 \tag{7.13c}$$

One value of $\Lambda_{ij}^{(k)}(0)$ will be specified by integrating a selected experimental measurement of $R_{ij}(\xi_k;0)$ over ξ_k. Note that this is not a universal specification but must be done for each individual flow analyzed.

The α_{ij}, β_{ij} and γ_{ij} parameters, which are τ dependent serve to satisfy continuity in the integral sense discussed above. Q_{ij} will have zero crossings dependent upon the determination of the α_{ij}, β_{ij} and γ_{ij} functions, (as well as on $\sigma_{ij}c_k$).

7.4. Model Limitations

The most serious limitation of the model is its restriction to only the integral scales of turbulence. Thus, although the behavior of the energy containing eddies can be represented by the model, there is no explicit dependence on the turbulent dissipation scale, or microscale. Therefore, we must expect that the chosen double-velocity correlation function will not provide a good representation of data in the region of small spatial separation. This can be remedied by adding additional terms to our model which are scaled by $\lambda_{ij}^{(k)}$ rather than by $\Lambda_{ij}^{(k)}$ where $\lambda_{ij}^{(k)}$ is the directional microscale

for each Q_{ij} . Indeed, the ideal representation would be an integral function which incorporated the entire spectrum of scales which occur in the flow and which account for the behavior of the turbulence in wavenumber space. However, for this initial study the analysis will be based on an integral scale model only. Note that better agreement could be achieved near $\xi_k = 0$ by using an $\exp(-|\xi_k|)$ type dependence. A Gaussian was chosen here in anticipation of adding the microscale dependence once our concept has been proven valid.

8. MOMENTS OF GOVERNING EQUATIONS

8.1. Convention for Nondimensionalization

Equations (7.10) are now substituted into Eq. (5.5) and integral moments are taken with respect to the coordinate directions. Before doing this, it is convenient to nondimensionalize the variables by a length scale $q/(dU_1/dx_2)$, a time scale $(dU_1/dx_2)^{-1}$, and a velocity scale q .

The definition of A , Eq. (6.7) is carried forward and we define

$$N = \frac{v_c}{A} \tag{8.1}$$

The parameters used to nondimensionalize the variables of the analysis are given below in Table 2.

With the understanding that all variables appearing henceforth are nondimensionalized unless noted otherwise, the model equations governing Q_{ij} appear as

$$\frac{\partial Q_{ij}}{\partial \tau} + (U + \xi_2)\frac{\partial Q_{ij}}{\partial \xi_1} = -(\delta_{i1}Q_{2j} + \delta_{j1}Q_{i2}) - A[Q_{ij} - \tfrac{1}{3}\delta_{ij}Q_{\ell\ell}]$$
$$+ N\nabla^2 Q_{ij} - 2bAQ_{ij} \tag{8.2}$$

where

$$Q_{ij} = u_{ij}R_{ij}g_{ij}(\tau) \qquad \text{(no summation)} \tag{8.3}$$

and

$$R_{ij} = [1 - \alpha_{ij}(\xi_1 - U\tau)^2 - \beta_{ij}\xi_2^2 - \gamma_{ij}\xi_3^2 - \mu_{ij}(\xi_1 - U\tau)\xi_2]$$
$$\cdot \exp\left[-\frac{1}{\sigma_{ij}^2}\left\{\frac{1}{c_1^2}(\xi_1 - U\tau)^2 + \frac{\xi_2^2}{c_2^2} + \frac{\xi_3^2}{c_3^2}\right\}\right] \text{(no summation)} \tag{8.4}$$

TABLE 2

Nondimensionalization of Variables

Variable	Nondimensionalized by:	Nondimensional notation
Q_{ij}	q^2	Q_{ij}
U_{1_o}	q	U
$(\overline{u_i u_j})_o$	q^2	u_{ij}
α_{ij}	$[q/(dU_1/dx_2)]^2$	α_{ij}
β_{ij}	↓	β_{ij}
γ_{ij}	↓	γ_{ij}
$\Lambda_{ij}^{(k)}$	$q/(dU_1/dx_2)$	$\Lambda_{ij}^{(k)}$
ξ_k	↓	ξ_k
σ_{ij}	↓	σ_{ij}
τ	$(dU_1/dx_2)^{-1}$	τ

8.2. Summation Convention

Since it will be convenient from this point on to discard the summation convention when denoting the velocity correlation subscripts i and j and the direction subscript k , these will be reserved exclusively for these purposes. Therefore, whenever i , j and k appear as subscripts, no summation will be implied. Other subscripts denote the summation convention as usual.

8.3. Moment Function

There are no set rules regarding which moments will give an optimum result when using this method to obtain solutions to any given problem. Given the number of unknowns and the governing differential equations determines the number of moments that must be taken to obtain a sufficient set for solution. The philosophy used during this analysis was to develop a family of rational moment functions and to avoid a partial application of any subset of this function. For instance, if the function were ξ_k , k would have to range from 1 to 3, and not just from 1 to 2, etc.

The function chosen can be written as $\Pi(m_k)$ where the argument denotes the product of coordinates of subscript k each to the power m_k , k ranging from 1 to 3.

$$\Pi(m_k) \equiv \Pi(m_1,m_2,m_3) \equiv \xi_1^{m_1}\xi_2^{m_2}\xi_3^{m_3} \tag{8.5}$$

thus for example

$$\Pi(0,1,2) \equiv \xi_1^0\xi_2^1\xi_3^2 \equiv \xi_2\xi_3^2$$

The integrated moment of Q_{ij} with weighting function $\Pi(m_k)$ is then defined as

$$I_{ij}^{(m_k)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Pi(m_k)Q_{ij}d\xi_1 d\xi_2 d\xi_3 \tag{8.6}$$

The integral moment of Q_{ij} with weighting function, $\xi_2\xi_3^2$ is therefore expressed $I_{ij}^{(0,1,2)}$.

Other integral moments that will be required are defined as

$$C_{ij}^{(m_k)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Pi(m_k)[U + \xi_2] \frac{\partial Q_{ij}}{\partial \xi_1} d\xi_1 d\xi_2 d\xi_3 \tag{8.7}$$

$$D_{ij}^{(m_k)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Pi(m_k)\nabla^2 Q_{ij}\, d\xi_1 d\xi_2 d\xi_3 \tag{8.8}$$

8.5. Family of Moment Equations

Taking the moments of Eqs. (8.2) with respect to the weighting function $\Pi(m_k)$ then leads to a family of equations of the form

$$\begin{aligned} \frac{d}{d\tau}\left[I_{ij}^{(m_k)}\right] + C_{ij}^{(m_k)} &= -\left[\delta_{i1} I_{2j}^{(m_k)} + \delta_{j1} I_{i2}^{(m_k)}\right] \\ &\quad - A\left[I_{ij}^{(m_k)} - \frac{1}{3}\delta_{ij} I_{\ell\ell}^{(m_k)}\right] \\ &\quad + N D_{ij}^{(m_k)} - 2bA I_{ij}^{(m_k)} \end{aligned} \tag{8.9}$$

Integration by parts may be used to evaluate $C_{ij}^{(m_k)}$, while Green's theorem is used to determine $D_{ij}^{(m_k)}$ rather simply. Over volume V

$$\begin{aligned} &\iiint_V \left[\Pi(m_k)\nabla^2 Q_{ij} - Q_{ij}\nabla^2\,\Pi(m_k)\right] d\xi_1 d\xi_2 d\xi_3 \\ &\qquad = \oiint_S \vec{n}\cdot\left[\Pi(m_k)\nabla Q_{ij} - Q_{ij}\nabla\,\Pi(m_k)\right] dS \end{aligned} \tag{8.10}$$

where S is the area bounding the volume of integration V and n is the outward pointing unit normal to S. Letting V go to infinity, the bounding integrals vanish since Q_{ij} approaches zero exponentially. Then

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Pi(m_k)\nabla^2 Q_{ij}\, d\xi_1 d\xi_2 d\xi_3 \\ &\qquad = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q_{ij}\nabla^2\,\Pi(m_k)\, d\xi_1 d\xi_2 d\xi_3 \end{aligned} \tag{8.11}$$

Starting with $m_k = (0,0,0)$ the moments can be evaluated in increasing order.

The lowest or zero moment $m_k = (0,0,0)$ is related to the anisotropic scales $\Lambda_{ij}^{(k)}$, where the moments are taken along the coordinates, see Eqs. (7.12). The first moments in each coordinate direction $m_k = (1,0,0)$; $(0,1,0)$; $(0,0,1)$ are zero since the turbulence is homogeneous and these are odd moments. The first mixed moments, $m_k = (1,1,0)$; $(1,0,1)$; $(0,1,1)$ make a contribution due to the skew symmetry produced by shear in the (ξ_1,ξ_2) plane (manifested in the correlation function by the presence of μ_{ij}). The family of moments required for this analysis ends with the second moments $m_k = (2,0,0)$; $(0,2,0)$; $(0,0,2)$ giving a sufficient set of equations to provide solutions for all parameters.

Before evaluating the integrals defining the moment functions some definitions are made for convenience. Let

$$\hat{\alpha}_{ij} = \alpha_{ij}\sigma_{ij}^2 c_1^2 \tag{8.12a}$$

$$\hat{\beta}_{ij} = \beta_{ij}\sigma_{ij}^2 c_2^2 \tag{8.12b}$$

$$\hat{\gamma}_{ij} = \gamma_{ij}\sigma_{ij}^2 c_3^2 \tag{8.12c}$$

$$\hat{\mu}_{ij} = \mu_{ij}\sigma_{ij}^2 c_1 c_2 \tag{8.12d}$$

The correlation function, Eq. (8.3), is now substituted into the integrals defining $\Lambda_{ij}^{(k)}$, $I_{ij}^{(m_k)}$, $C_{ij}^{(m_k)}$ and $D_{ij}^{(m_k)}$ to relate them to the correlation function parameters. Note that these integrals are to be evaluated for $U = 0$ since the parameters are being calculated in a convected coordinate system moving with the mean flow.

<u>$m_k = (0,0,0)$</u> - The specialized form of the zero moments are the anisotropic scales $\Lambda_{ij}^{(k)}$

$$\Lambda_{ij}^{(1)} = \frac{\sqrt{\pi}}{4} c_1\sigma_{ij}[2 - \hat{\alpha}_{ij}] \tag{8.13a}$$

$$\Lambda_{ij}^{(2)} = \frac{\sqrt{\pi}}{4} c_2 \sigma_{ij} [2 - \hat{\beta}_{ij}] \tag{8.13b}$$

$$\Lambda_{ij}^{(3)} = \frac{\sqrt{\pi}}{4} c_3 \sigma_{ij} [2 - \hat{\gamma}_{ij}] \tag{8.13c}$$

<u>m_k = (1,0,0) ; (0,1,0) ; (0,0,1)</u> - As noted above the first moments are zero.

<u>m_k = (1,1,0) ; (1,0,1) ; (0,1,1)</u> - Only the first of these three mixed moments is non-zero and results in

$$I_{ij}^{(1,1,0)} = \frac{\pi^{3/2}}{4} u_{ij} g_{ij} \sigma_{ij}^5 c_1 c_2 c_3 \hat{\mu}_{ij} \tag{8.14}$$

The associated values of $C_{ij}^{(m_k)}$ and $D_{ij}^{(m_k)}$ are

$$C_{ij}^{(1,1,0)} = \frac{\pi^{3/2}}{2} u_{ij} g_{ij} \sigma_{ij}^5 c_1 c_2^3 c_3 \, [1 - \tfrac{1}{2}(\hat{\alpha}_{ij} + 3\hat{\beta}_{ij} + \hat{\gamma}_{ij})] \tag{8.15}$$

$$D_{ij}^{(1,1,0)} = 0 \tag{8.16}$$

<u>m_k = (2,0,0) ; (0,2,0) ; (0,0,2)</u>

$$I_{ij}^{(2,0,0)} = \frac{\pi^{3/2}}{2} u_{ij} g_{ij} \sigma_{ij}^5 c_1^3 c_2 c_3 \, [1 - \tfrac{1}{2}(3\hat{\alpha}_{ij} + \hat{\beta}_{ij} + \hat{\gamma}_{ij})] \tag{8.17a}$$

$$I_{ij}^{(0,2,0)} = \frac{\pi^{3/2}}{2} u_{ij} g_{ij} \sigma_{ij}^5 c_1 c_2^3 c_3 \, [1 - \tfrac{1}{2}(\hat{\alpha}_{ij} + 3\hat{\beta}_{ij} + \hat{\gamma}_{ij})] \tag{8.17b}$$

$$I_{ij}^{(0,0,2)} = \frac{\pi^{3/2}}{2} u_{ij} g_{ij} \sigma_{ij}^5 c_1 c_2 c_3^3 \, [1 - \tfrac{1}{2}(\hat{\alpha}_{ij} + \hat{\beta}_{ij} + 3\hat{\gamma}_{ij})] \tag{8.17c}$$

$$C_{ij}^{(2,0,0)} = \frac{\pi^{3/2}}{2} u_{ij} g_{ij} \sigma_{ij}^5 c_1 c_2 c_3 \hat{\mu}_{ij} \delta_{1k} \tag{8.18a}$$

$$C_{ij}^{(0,2,0)} = C_{ij}^{(0,0,2)} = 0 \tag{8.18b,c}$$

$$D_{ij}^{(2,0,0)} = D_{ij}^{(0,2,0)} = D_{ij}^{(0,0,2)} \tag{8.19a,b,c}$$

$$= 2\pi^{3/2} u_{ij} g_{ij} \sigma_{ij}^3 c_1 c_2 c_3 \quad [1 - \tfrac{1}{2}(\hat{\alpha}_{ij} + \hat{\beta}_{ij} + \hat{\gamma}_{ij})]$$

The mixed moment (1,1,0) provides one set of equations in the 11,22,33,12 components. The second moments provide three equations for each ij component corresponding to the three coordinate directional moments. Note that $C_{ij}^{(2,0,0)} = 2\sigma_{1k} I^{(1,1,0)}$ and $C_{ij}^{(1,1,0)} = I_{ij}^{(0,2,0)}$, coupling the mixed and second moments.

8.6. Integral Continuity Constraints

Substitution of the correlation function, Eq. (8.3) and Eq. (8.4), into the integral constraints, Eq. (7.12), yields the following five equations

$$\hat{\beta}_{11} + \hat{\gamma}_{11} = 2 \tag{8.20a}$$

$$\hat{\alpha}_{22} + \hat{\gamma}_{22} = 2 \tag{8.20b}$$

$$\hat{\alpha}_{33} + \hat{\beta}_{33} = 2 \tag{8.20c}$$

$$\hat{\alpha}_{12} + \hat{\gamma}_{12} = 2 \tag{8.20d}$$

$$\hat{\beta}_{12} + \hat{\gamma}_{12} = 2 \tag{8.20e}$$

8.7. Method of Solution

The integral moment equations, the continuity constraints and only one of the integral scales are sufficient to define the correlation function parameters by integrating Eqs. (8.9). At $\tau = 0$, $\partial Q_{ij}/\partial\tau = 0$ is the initial condition required. Differentiation of the correlation function with respect to τ yields the conditions that $\dot{\alpha}_{ij}(o) = 0$, $\dot{\beta}_{ij}(o) = 0$, etc., for this initial condition to

be valid universally. With these conditions Eqs. (8.9) become a set of nonlinear algebraic equations in the correlation parameters at $\tau = 0$ which may be solved to determine their initial values. If it is assumed that the acoustic sources are compact then a solution derived by expansion of the variables about $\tau = 0$ can be determined, which has the advantage of providing an analytical solution for the higher derivatives in τ. Since the fourth derivative with respect to τ is required to evaluate the acoustic integral, this approach will avoid numerical evaluation of the derivatives.

Since the first derivatives are zero at $\tau = 0$ the correlation parameters which are chosen to be functions of time in this simplified analysis are expanded in series in τ as follows

$$\hat{\alpha}_{ij} = \hat{\alpha}_{ij_o} + \hat{\alpha}_{ij_2}\tau^2 + \hat{\alpha}_{ij_4}\tau^4 + \cdots \tag{8.21a}$$

$$\hat{\beta}_{ij} = \hat{\beta}_{ij_o} + \hat{\beta}_{ij_2}\tau^2 + \hat{\beta}_{ij_4}\tau^4 + \ldots \tag{8.21b}$$

$$\hat{\gamma}_{ij} = \hat{\gamma}_{ij_o} + \hat{\gamma}_{ij_2}\tau^2 + \hat{\gamma}_{ij_4}\tau^4 + \ldots \tag{8.21c}$$

$$g_{ij} = 1 + g_{ij_2}\tau^2 + g_{ij_4}\tau^4 + \ldots \tag{8.21d}$$

The decorrelation function b is τ dependent, as it must be for g_{ij} to vary from its initial value. Expressing b as a series in τ/τ_D, where τ_D is the characteristic turbulence time Λ/q, the nondimensional expansion is

$$b = b_o + b_1A\tau + b_2A^2\tau^2 + b_3A^3\tau^3 + b_4A^4\tau^4 + \ldots \tag{8.22}$$

where b_n are specified constants. Nothing is known about this function at the present time and definition of the physics and form of this parameter is put off to a following study. b_n will be chosen by matching the results of the present analysis to the distribution of measured acoustic data.

Substitution of the expansions, Eq. (8.21) and Eq. (8.22), into the moment functions, Eq. (8.14) to Eq. (8.19), yields series expressions for $I_{ij}^{(m_k)}$, $C_{ij}^{(m_k)}$ and $D_{ij}^{(m_k)}$ in powers of τ with coefficients which are functions of $\hat{\alpha}_{ij}$, $\hat{\rho}_{ij}$, etc.

$$I_{ij}^{(m_k)} = I_{ij_o}^{(m_k)} + I_{ij_2}^{(m_k)}\tau^2 + I_{ij_3}^{(m_k)}\tau^3 + I_{ij_4}^{(m_k)}\tau^4 + \ldots \tag{8.23a}$$

$$C_{ij}^{(m_k)} = C_{ij_o}^{(m_k)} + C_{ij_2}^{(m_k)}\tau^2 + C_{ij_3}^{(m_k)}\tau^3 + C_{ij_4}^{(m_k)}\tau^4 + \ldots \tag{8.23b}$$

$$D_{ij}^{(m_k)} = D_{ij_o}^{(m_k)} + D_{ij_2}^{(m_k)}\tau^2 + D_{ij_3}^{(m_k)}\tau^3 + D_{ij_4}^{(m_k)}\tau^4 + \ldots \tag{8.23c}$$

Separation of Eqs. (8.9) into terms in powers of τ then provides a set which may be solved for successively higher coefficients.

$\underline{\tau^o:}$

$$C_{ij_o}^{(m_k)} + \left[\delta_{i1} I_{2j_o}^{(m_k)} + \delta_{j1} I_{i2}^{(m_k)}\right] + A\left[I_{ij_o}^{(m_k)} - \tfrac{1}{3}\delta_{ij} I_{\ell\ell_o}^{(m_k)}\right]$$
$$- ND_{ij_o}^{(m_k)} + 2b_o A I_{ij_o}^{(m_k)} = 0 \tag{8.24a}$$

$\underline{\tau^1:}$

$$I_{ij_2}^{(m_k)} = - b_1 A^2 I_{ij_o}^{(m_k)} \tag{8.24b}$$

$\underline{\tau^2:}$

$$3I_{ij_3}^{(m_k)} = - C_{ij_2}^{(m_k)} - \left[\delta_{i1} I_{2j_2}^{(m_k)} + \delta_{j1} I_{i2_2}^{(m_k)}\right] - A\left[I_{ij_2}^{(m_k)} - \tfrac{1}{3}\delta_{ij} I_{\ell\ell_2}^{(m_k)}\right]$$
$$+ ND_{ij_2}^{(m_k)} - 2A\left[b_o I_{ij_2}^{(m_k)} + b_2 A^2 I_{ij_o}^{(m_k)}\right] \tag{8.24c}$$

$\underline{\tau^3:}$

$$4I_{ij_4}^{(m_k)} = - C_{ij_3}^{(m_k)} - \left[\delta_{i1} I_{2j_3}^{(m_k)} + \delta_{j1} I_{i2_3}^{(m_k)}\right] - A\left[I_{ij_3}^{(m_k)} - \tfrac{1}{3}\delta_{ij} I_{\ell\ell_3}^{(m_k)}\right]$$
$$+ ND_{ij_3}^{(m_k)} - 2A\left[b_o I_{ij_3}^{(m_k)} + b_1 A I_{ij_2}^{(m_k)} + b_3 A^3 I_{ij_o}^{(m_k)}\right] \tag{8.24d}$$

Solving the nonlinear algebraic set of zero-order equations, Eqs. (8.24a), provides the initial conditions for the correlation parameters at $\tau = 0$. The model constants that must be specified are A, b_o, u_{ij} and v_c. The solution provides α_{ij_o}, β_{ij_o}, γ_{ij_o}, μ_{ij_o}, σ_{ij_o} and c_{k_o}. The initial value of $g_{ij}^o(\tau)$ is specified as $g_{ij}(o) = g_{ij_o} = 1.0$. For now, we assume σ_{ij}, μ_{ij} and c_k do not vary with separation time, affording considerable simplification without undue loss in generality.

Since the higher-order coefficient equations are linear in those of lower order, Eqs. (8.24) provide the series expansion coefficients needed to define the separation-time derivatives at $\tau = 0$ using elementary matrix algebra.

9. RESULTS AND COMPARISON WITH TURBULENT SHEAR FLOW DATA

Tables 3 and 4 present the zero-order coefficient solutions to Eq. (8.24a) for model constant values $A = \sqrt{2}$, $u_{11} = 1/2$, $u_{22} = 1/4$, $u_{33} = 1/4$, $u_{12} = -1/(4\sqrt{2})$, $v_c = 0.3$ and three values of b_o: 0.0, 0.02 and 0.05. Figures 3 and 4 are plots of the correlation function R_{ij} for $b_o = 0.0$ and 0.05, respectively. Comparison of these figures with measured two-point, one-time turbulence correlations makes possible an evaluation of the qualitative aspects of the results. Such measurements in a two-dimensional constant shear flow have been made and reported by Champagne, Harris and Corrsin[17], Harris, Graham and Corrsin[18], and Rose[28]. Figure 5 presents measured values of R_{11} and R_{12} measured along the coordinate directions at $\tau = 0$ as measured by Harris, et al.[18] The directional behavior of these correlations are typical of those measured by Champagne, et al.[17] and Rose et al.[28] The measurements in the ξ_3 directions, normal to the mean flow and velocity gradients, decrease most rapidly with separation distance and exhibit extensive regions of negative correlation. Those in the flow direction, ξ_1, are slowest to decorrelate. These characteristics are present in general in the theoretical predictions, although the trends are somewhat exaggerated. Since $R_{ij}(\xi;0) \leq R_{ij}(0,0)$ in a physical flow, the theoretical behavior of $R_{33}(0,0,\xi_3;0)$ in both Figures 3 and 4 is the most serious discrepancy in the model. It should be remembered, however, that the present model is an integral representation and as such may exhibit local inconsistencies with little serious effect in its portrayal of overall behavior. Including the effects of a turbulent microscale explicitly in the correlation function would probably eliminate the undesirable overshoot in $R_{33}(0,0,\xi_3;0)$. Examination of the model also reveals that employing a directionally dependent value of v_c could reduce the negative overshoot in $R_{ij}(0,0,\xi_3;0)$ while increasing the rate of decorrelation in $R_{ij}(\xi_1,0,0;0)$ with separation distance, thereby improving comparison between measurement and theory. This was not done for the present until more empirical evidence can be found to justify this extension.

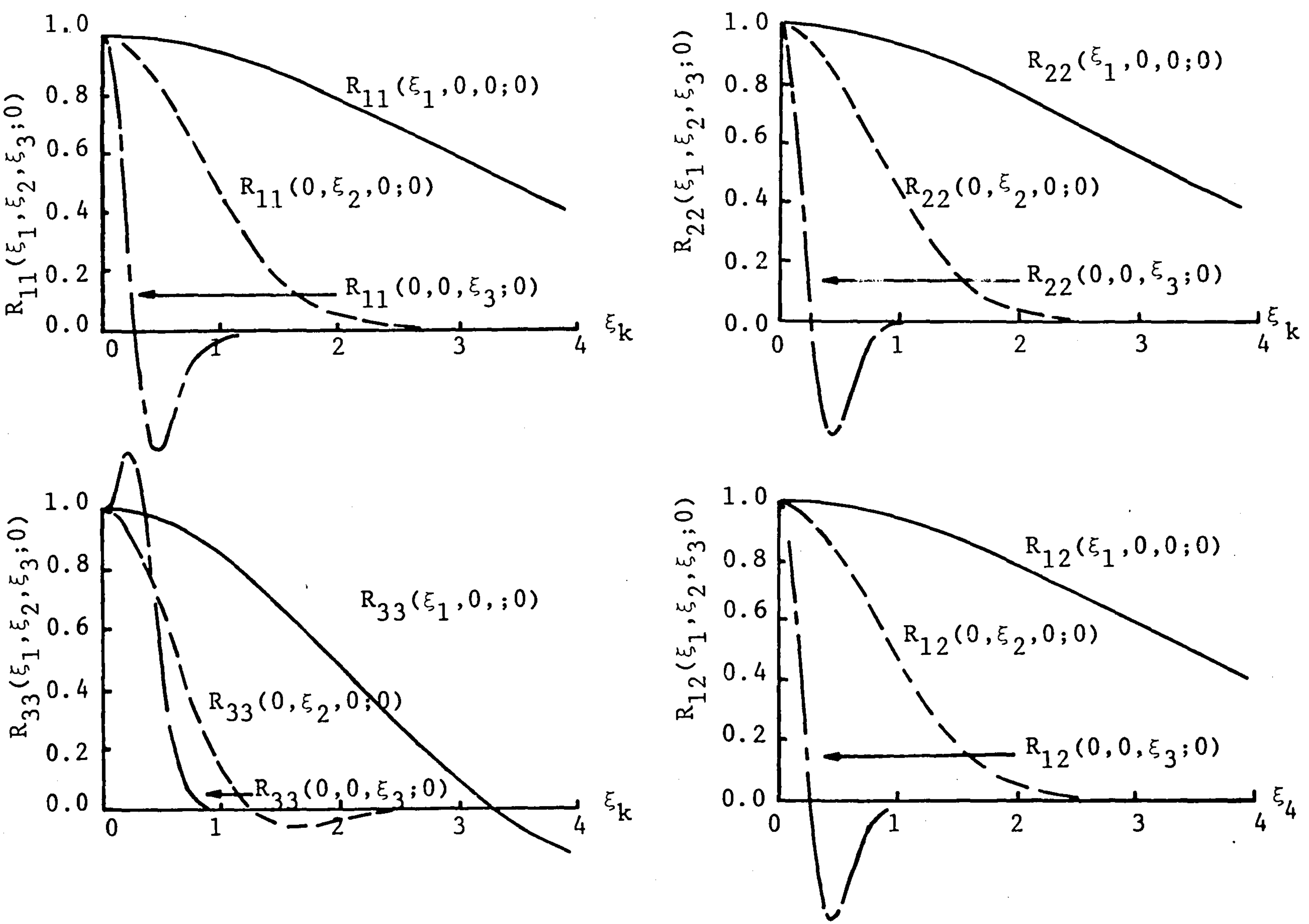


Figure 3. Theoretical two-point correlations at $\tau = 0$ for $ij = 11,22,33,12$ and $b = 0.0$

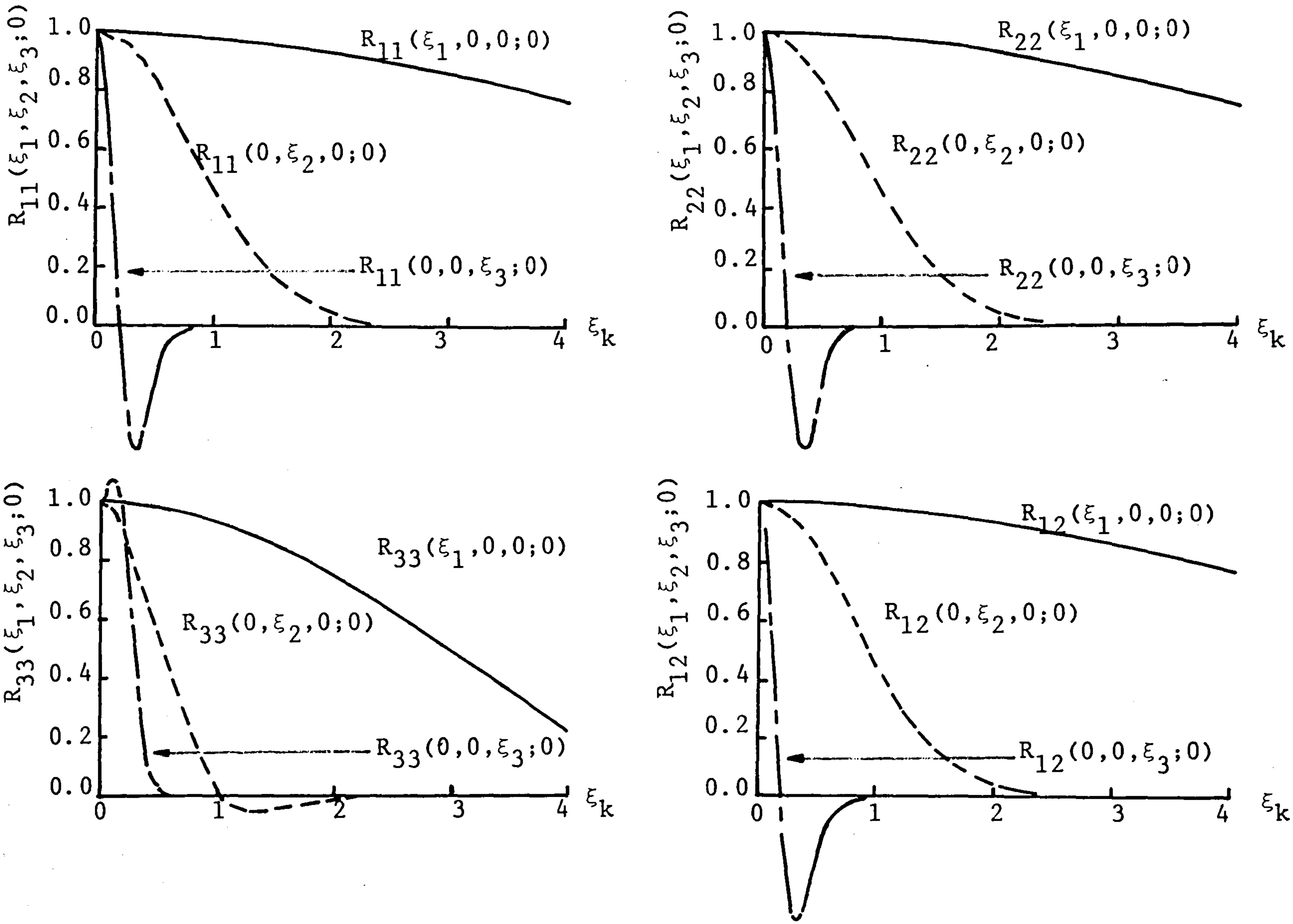
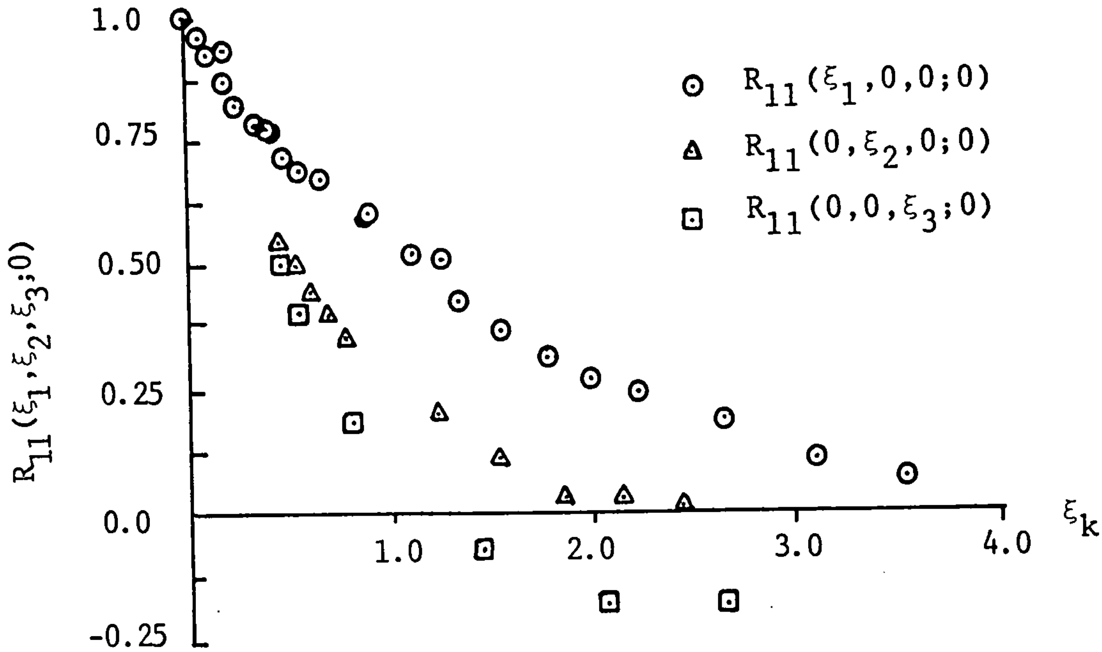


Figure 4. Theoretical two-point correlations at $\tau = 0$ for $ij = 11, 22, 33, 12$ and $b = 0.05$



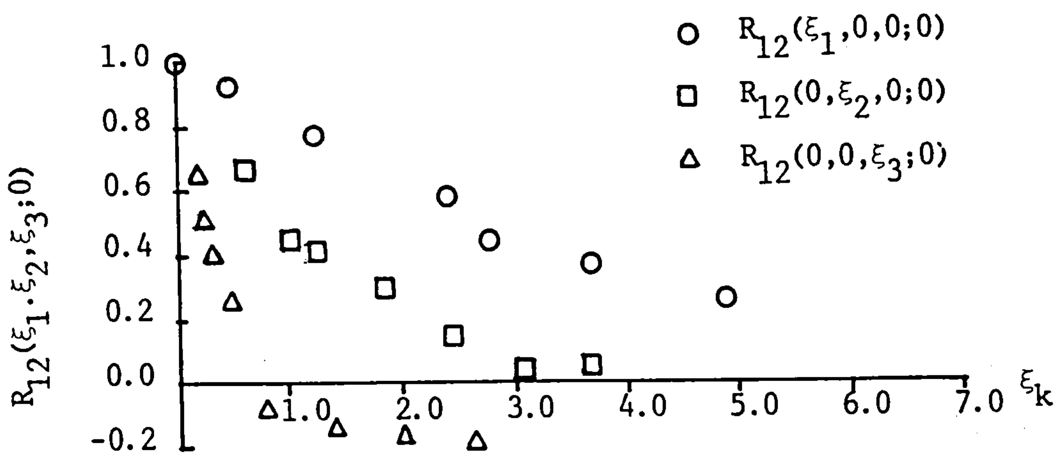


Figure 5. Measured two-point correlations at $\tau = 0$, from Harris, Graham and Corrsin.[18] $q = 90.5$ cm/sec. $dU_1/dx_2 = 44\ \text{sec}^{-1}$

TABLE 3

Trial-Function Coefficients at Zero Time Separation

ij	$\hat{\alpha}_{ij_o}$	$\hat{\beta}_{ij_o}$	$\hat{\gamma}_{ij_o}$	$\hat{\mu}_{ij_o}$	$\hat{\sigma}_{ij_o}$
			b = 0.0		
11	0.1227	0.1169	1.8831	-0.0990	1.1985
22	0.1746	0.1151	1.8254	-0.1495	1.1675
33	1.308	0.6916	-1.9082	-0.2216	1.0254
12	0.1186	0.1186	1.8814	-0.1028	1.1959
			b = 0.02		
11	0.1068	0.1028	1.8972	-0.0784	1.1896
22	0.1500	0.1015	1.8500	-0.1086	1.1687
33	1.3277	0.6723	-1.7699	-0.1730	0.9739
12	0.0037	0.0737	1.6768	-0.0237	1.1882
			b = 0.05		
11	0.0743	0.0721	1.9279	-0.0386	1.1706
22	0.1006	0.0718	1.8994	-0.0493	1.1624
33	1.3726	0.6274	-1.5060	-0.0572	0.8522
12	0.0728	0.0728	1.9272	-0.0382	1.1700

TABLE 4

Scale Factors at Zero Time Separation

b(o) / k	0.0	0.02	0.05
1	3.6313	4.4728	6.8162
2	1.0000	1.0000	1.0000
3	0.3028	0.2830	0.2358

10. RADIATED ACOUSTIC POWER BASED ON TURBULENT CORRELATION FUNCTION

From Eq. (3.8) the acoustic power radiated in direction (ψ,ϕ) from a unit volume element at $\vec{y}$ is, in the Proudman formulation and in dimensional form

$$P(\psi,\phi,\vec{y}) = \frac{\rho_o}{16\pi^2 c_o^5 x^2}\int_\infty \frac{\partial^4}{\partial\tau^4}\,\overline{v_x^2 v_x'^2}\, d\vec{\xi} \tag{10.1}$$

where v_x and v_x' are the components of total velocity at y' and y'' in the direction of $\vec{x}$, see Figure 1. Since $\vec{x}$ and $\vec{r}$ are nearly parallel for $\vec{x}$ in the acoustic far field, we shall approximate $\vec{x}$ by $\vec{r}$. Let the total velocity at $\vec{\xi} = 0$ be $\vec{v}$ and at $\vec{\xi}$, $\vec{v}'$. The mean flow is along ξ_1 so that the components of total velocity are

$$v_i = u_i + \delta_{1k}U \tag{10.2a}$$

$$v_i' = u_i' + \delta_{1k}U' \tag{10.2b}$$

where u_i and u_i' are the fluctuating components. The velocity components v_x and v_x' in terms of v_i and v_i' are, to a first approximation*

$$v_x = v_1 \cos\psi + v_2 \sin\psi \cos\phi + v_3 \sin\psi \sin\phi \qquad (10.3a)$$

$$v_x' = v_1 \cos\psi + v_2' \sin\psi \cos\phi + v_3' \sin\psi \sin\phi \qquad (10.3b)$$

The correlation $\overline{v_x^2 v_x'^2}$ is found by substituting Eq. (10.2) into Eq. (10.3); squaring Eq. (10.3a) and Eq. (10.3b) and forming their product and taking the ensemble average. The result is an expression containing terms in $\overline{u_i u_p u_j' u_q'}$, $UU'\overline{u_i u_j'}$, $U^2U'^2$, $U^2\overline{u_i}^2$, $U^2\overline{u_j'}^2$

$U\overline{u_i u_j'^2}$, and $U'\overline{u_i' u_j}^2$. The first two of these will contribute to the flow noise. The following three are constant with τ and will vanish upon differentiation. The last two terms are triple-velocity correlations. For homogeneous, stationary, turbulence, these are odd functions by invariance with reflection of coordinates. Change of sign of coordinates must be accompanied by change of sign of separation time. Since we have assumed the correlation function may be expressed as the product of separation space and separation time variables, the triple-velocity correlations will integrate to zero over separation space.

In order to provide a tractable analysis for the fourth-order correlations, normal joint probability of u_i and u_j is assumed. This is done based on Batchelors[27] argument that the part of the joint probability distribution of the velocities associated with the energy containing eddies is approximately normal at a fixed time and at points sufficiently separated in space. This assumption has been used by numerous investigators.[1,6,22,23,25,26,27,29] Goldstein and Rosenbaum[26] extended this argument to time separation by arguing that the correlations will be subject to even more random influences from the neighboring flow when they are separated in time. Thus, their joint probability should be even closer to normal.

Using the assumption of normal joint probability the fourth-order velocity correlations may be written (see Batchelor[27])

---

* The effect of $\vec{\xi}$ on the components of v_i and v_i' in the direction of $\vec{x}$ is of order $\xi/x \ll 1$ and will be neglected in this analysis.

$$\overline{u_i u_p u'_j u'_q} = (\overline{u_i u_p})(\overline{u'_j u'}) + (\overline{u_i u'_j})(\overline{u_p u'_q}) + (\overline{u_i u'_q})(\overline{u_p u'_j})$$

$$= Q_{ip}(0)Q_{jq}(0) + Q_{ij}Q_{pq} + Q_{iq}Q_{pj} \qquad (10.4)$$

The first term on the right will vanish upon differentiation with respect to τ .

The other nonvanishing term is of the form $UU'\overline{u_i u'_j}$, where $U' = U + \xi_2$ nondimensionally. For simplicity the variation in U with ξ_2 is dropped and only the leading term in U^2 is retained. This leaves terms of the form $U^2\overline{u_i u'_j}$.

The noise associated with the fourth-order turbulent correlations has been designated self-noise, while that generated by terms of the form $U^2\overline{u_i u'_j}$ is known as shear noise. These contributions will be written separately in what follows.

$P(\psi,\phi,\vec{y})$, the sound power radiated along $\vec{x}$ in the direction (ψ,ϕ) per unit volume of shear flow per unit area at x may be expressed as

$$\hat{P}(\psi,\phi,\vec{y}) = \frac{1}{16\pi^2}\frac{L_m}{U_m}\frac{dU_1}{dx_2}\left[\frac{q(\vec{y})}{U_m}\right]^7 \left\{U^2(\vec{y})\sum_{ij} A_{ij}u_{ij}\frac{\partial^4}{\partial\tau^4}(g_{ij}W_{ij}) + \sum_{ijpq} B_{ijpq}u_{ij}u_{pq}\frac{\partial^4}{\partial\tau^4}\left(g_{ij}g_{pq}W_{ijpq}\right)\right\} \qquad (10.5)$$

Although not explicitly stated, u_{ij} and u_{pq} are functions of absolute space when a real flow is treated using the "locally homogeneous" approximation.

A_{ij} and B_{ijpq} are functions of ψ and ϕ as derived by expansion of $\overline{v_x^2 v_x'^2}$ The summation sign indicates summation over all correlations that contribute to the sound power, $P(\psi,\phi,y)$ is nondimensionalized as follows

$$\hat{P}(\psi,\phi,\vec{y}) = \frac{c_o^5 x^2 L_m P(\psi,\phi,\vec{y})}{\rho_o U_m^8} \qquad (10.6)$$

W_{ij} and W_{ijpq} are the integrals of R_{ij} and $R_{ij}R_{pq}$ over separation space

$$W_{ij} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{ij}d\xi_1 d\xi_2 d\xi_3 \tag{10.7a}$$

$$W_{ijpq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{ij}R_{pq}d\xi_1 d\xi_2 d\xi_3 \tag{10.7b}$$

U_m is a reference velocity and L_m a reference length. Neither of these is a function of $\vec{y}$.

The values of A_{ij} and B_{ijpq} derived by expanding $\overline{v_x^2 v_x'^2}$ for the correlations which have been included in this analysis and which contribute to the sound power are shown in Tables 5a and 5b. The coefficients in these tables include the permutations of the indices of the correlations to include all contributing terms.

If the shear flow is axisymmetric the sound power may be averaged over ϕ , since the nonaxisymmetric power from individual volume elements of the flow will mutually cancel on a time average basis.[22] Define

$$\overline{A}_{ij} = \frac{1}{2\pi}\int_{0}^{2\pi} A_{ij}d\phi \tag{10.8a}$$

$$\overline{B}_{ijpq} = \frac{1}{2\pi}\int_{0}^{2\pi} B_{ijpq}d\phi \tag{10.8b}$$

The results of these averages are shown in Tables 6a and 6b. The ϕ-averaged sound power is expressed as

$$P(\psi,\vec{y}) = \frac{1}{16\pi^2}\frac{L_m}{U_m}\frac{dU_1}{dx_2}\left[\frac{q(\vec{y})}{U_m}\right]^7 \left\{ U^2(\vec{y})\sum_{ij}\overline{A}_{ij}u_{ij}\frac{\partial^4}{\partial\tau^4}(g_{ij}W_{ij}) + \sum_{ijpq}\overline{B}_{ijpq}u_{ij}u_{pq}\frac{\partial^4}{\partial\tau^4}\left(g_{ij}g_{pq}W_{ijpq}\right)\right\} \tag{10.9}$$

Sound Directivity of ij and ijpq Components

TABLE 5a

ij	A_{ij}
11	$4 \cos^4 \psi$
22	$4 \cos^2 \psi \sin^2 \psi \cos^2 \phi$
33	$4 \cos^2 \psi \sin^2 \psi \sin^2 \phi$
12	$8 \cos^3 \psi \sin \psi \cos \phi$

TABLE 5b

ijpq	ijpq
1111	$2 \cos^4 \psi$
2222	$2 \sin^4 \psi \cos^4 \phi$
3333	$2 \sin^4 \psi \sin^4 \phi$
1212	$8 \cos^2 \psi \sin^2 \psi \cos^2 \phi$
1122	$4 \cos^2 \psi \sin^2 \psi \cos^2 \phi$
1133	$4 \sin^2 \psi \cos^2 \psi \sin^2 \phi$
2233	$4 \sin^4 \psi \sin^2 \phi \cos^2 \phi$
1112	$8 \sin \psi \cos^3 \psi \cos \phi$
2212	$8 \sin^3 \psi \cos \psi \cos^3 \phi$
3312	$8 \sin^3 \psi \cos \psi \sin^2 \phi \cos \phi$

ϕ-averaged Sound Directivity of ij and ijpq Components

TABLE 6a

ij	$\bar{A}_{ij}$
11	$4 \cos^4 \psi$
22	$2 \cos^2 \psi \sin^2 \psi$
33	$2 \cos^2 \psi \sin^2 \psi$
12	0

TABLE 6b

ijpq	$\bar{B}_{ijpq}$
1111	$2 \cos^4 \psi$
2222	$3/4 \sin^4 \psi$
3333	$3/4 \sin^4 \psi$
1212	$4 \cos^2 \psi \sin^2 \psi$
1122	$2 \cos^2 \psi \sin^2 \psi$
1133	$2 \cos^2 \psi \sin^2 \psi$
2233	$\frac{1}{2} \sin^4 \psi$
1112	0
2212	0
3312	0

Integration of R_{ij} and $R_{ij}R_{pq}$ over the separation volume yields

$$W_{ij} = \pi^{3/2} c_1 c_2 c_3 \sigma_{ij}^3 [1 - \tfrac{1}{2}(\alpha_{ij} + \beta_{ij} + \gamma_{ij})] \tag{10.10a}$$

$$\begin{aligned} W_{ijpq} = \pi^{3/2} c_1 c_2 c_3 \kappa_{ijpq}^3 [1 &- \tfrac{1}{2}\kappa_{ijpq}^2 \{(\hat{\alpha}_{ij} + \hat{\alpha}_{pq}) \\ &+ (\hat{\beta}_{ij} + \hat{\beta}_{pq}) + (\hat{\gamma}_{ij} + \hat{\gamma}_{pq})\} \\ &+ \tfrac{1}{4}\kappa_{ijpq}^4 \{3(\hat{\alpha}_{ij}\hat{\alpha}_{pq} + \hat{\beta}_{ij}\hat{\beta}_{pq} + \hat{\gamma}_{ij}\hat{\gamma}_{pq}) \\ &+ (\hat{\alpha}_{ij}\hat{\beta}_{pq} + \hat{\alpha}_{pq}\hat{\beta}_{ij}) + (\hat{\alpha}_{ij}\hat{\gamma}_{pq} + \hat{\alpha}_{pq}\hat{\gamma}_{ij}) \\ &+ (\hat{\beta}_{ij}\hat{\gamma}_{pq} + \hat{\beta}_{pq}\hat{\gamma}_{ij}) + \hat{\mu}_{ij}\hat{\mu}_{pq}\}] \end{aligned} \tag{10.10b}$$

where

$$\kappa_{ijpq}^2 = \left[\frac{\sigma_{ij}^2 \sigma_{pq}^2}{\sigma_{ij}^2 + \sigma_{pq}^2}\right] \tag{10.11}$$

The fourth-derivative of these functions at $\tau = 0$ are obtained by substitution of the series expansions for the correlation parameters, Eqs. (8.21), taking the derivatives and setting $\tau = 0$. The results of these operations are

$$\frac{\partial^4}{\partial\tau^4}(g_{ij}W_{ij}) = 24\pi^{3/2} c_1 c_2 c_3 \sigma_{ij}^3 \left[g_{ij_4} S_{ij_o}^{(0)} S_{ij_4}^{(0)}\right] \tag{10.12a}$$

$$\frac{\partial^4}{\partial\tau^4}\left(g_{ij}g_{pq}W_{ijpq}\right) = 24\pi^{3/2}c_1c_2c_3\kappa^3_{ijpq}\ \Bigg[\left(1 + \frac{1}{4}\hat{\kappa}^4_{ijpq}\hat{\mu}_{ij}\hat{\mu}_{pq}\right)G_4$$

$$- \frac{1}{4}\hat{\kappa}^2_{ij}\left(<\hat{\alpha}_{ij}G> + <\hat{\beta}_{ij}G> + <\hat{\gamma}_{ij}G>\right)$$

$$- \frac{1}{2}\hat{\kappa}^2_{pq}\left(<\hat{\alpha}_{pq}G> + <\hat{\beta}_{pq}G> + <\hat{\gamma}_{pq}G>\right)$$

$$+ \frac{1}{4}\hat{\kappa}^4_{ijpq}\Bigg\{3\left(<\hat{\alpha}_{ij}\hat{\alpha}_{pq}G> + <\hat{\beta}_{ij}\hat{\beta}_{pq}G> + <\hat{\gamma}_{ij}\hat{\gamma}_{pq}G>\right)$$

$$+ <\hat{\alpha}_{ij}\hat{\beta}_{pq}G> + <\hat{\alpha}_{pq}\hat{\beta}_{ij}G> + <\hat{\alpha}_{ij}\hat{\gamma}_{pq}G>$$

$$+ <\hat{\alpha}_{pq}\hat{\gamma}_{ij}G> + <\hat{\beta}_{ij}\hat{\gamma}_{pq}G> + <\hat{\gamma}_{ij}\hat{\beta}_{pq}G>\Bigg\}\Bigg] \tag{10.12b}$$

where the various functions in these expressions are defined by

$$S_{ij_o} = \left[1 - \frac{1}{2}\left(\hat{\alpha}_{ij_o} + \hat{\beta}_{ij_o} + \hat{\gamma}_{ij_o}\right)\right] \tag{10.13a}$$

$$S_{ij_4} = -\frac{1}{2}\left(\hat{\alpha}_{ij_4} + \hat{\beta}_{ij_4} + \hat{\gamma}_{ij_4}\right) \tag{10.13b}$$

$$\hat{\kappa}^2_{ij} = \left[1 + (\sigma_{ij}/\sigma_{pq})^2\right]^{-1} \tag{10.13c}$$

$$\hat{\kappa}^2_{pq} = \left[1 + (\sigma_{pq}/\sigma_{ij})^2\right]^{-1} \tag{10.13d}$$

$$\hat{\kappa}^4_{ijpq} = \left[(\sigma_{pq}/\sigma_{ij})^2 + (\sigma_{ij}/\sigma_{pq})^2\right]^{-2} \tag{10.13e}$$

$$<X_{ij}G> = X_{ij_o}G_4 + X_{ij_2}G_2 + X_{ij_4} \tag{10.13f}$$

$$<Y_{pq}G> = Y_{pq_o}G_4 + Y_{pq_2}G_2 + Y_{pq_4} \tag{10.13g}$$

$$<X_{ij} Y_{pq} G> = X_{ij_o} Y_{pq_o} G_4 + \left(X_{ij_2} Y_{pq_o} + X_{ij_o} Y_{pq_2} \right) G_2$$

$$+ X_{ij_2} Y_{pq_2} + X_{ij_o} Y_{pq_4} + X_{ij_4} Y_{pq_o} \qquad (10.13h)$$

$$G_2 = g_{ij_2} + g_{pq_2} \qquad (10.13i)$$

$$G_4 = g_{ij_4} + g_{ij_2} g_{pq_2} + g_{pq_4} \qquad (10.13j)$$

X_{ij_n} takes on the values of the expansion coefficients $\hat{\alpha}_{ij_n}$, $\hat{\beta}_{ij_n}$ and $\hat{\gamma}_{ij_n}$ where appropriate, while Y_{pq_n} symbolizes $\hat{\alpha}_{pq_n}$, $\hat{\beta}_{pq_n}$ and $\hat{\gamma}_{pq_n}$.

The sound power intensity $I(\psi)$ for axisymmetric flow may now be found by integrating Eq. (10.9) over the volume of absolute space containing the turbulence. Since u_{ij}, u_{pq} and q^2 may be functions of $\vec{y}$ for "locally homogeneous" turbulence, it is more convenient to nondimensionalize the one-point, one-time correlations by the reference mean velocity U_m. Thus, $I(\psi)$ can be written

$$I(\psi) = \frac{1}{16\pi^2} \frac{\rho_o U_m^8}{c_o^5 x^2 L_m} \iiint_{V_F} \left(\frac{L_m}{U_m} \frac{dU(\vec{y})}{dx_2} \right)$$

$$\cdot \left\{ \left(\frac{U(\vec{y})}{U_m} \right)^2 \left(\frac{q(\vec{y})}{U_m} \right)^5 \sum_{ij} \bar{A}_{ij} \left[\frac{(u_i u_j)_o}{U_m^2} \right] \frac{\partial^4}{\partial \tau^4} (g_{ij} W_{ij}) \right. \qquad (10.14)$$

$$\left. + \left(\frac{q(\vec{y})}{U_m} \right)^7 \sum_{ijpq} \bar{B}_{ijpq} \left[\frac{(u_i u_j)_o (u_p u_q)_o}{U_m^4} \right] \frac{\partial^4}{\partial \tau^4} (g_{ij} g_{pq} W_{ijpq}) \right\} d\vec{y}$$

where the integral is taken over the flow volume V_f and $(\overline{u_i u_j})_o$, $(\overline{u_p u_q})_o$, q and the gradient dU_1/dx_2 may be functions of $\vec{y}$.

This completes the equations necessary to compute the sound power, given the one-point, one-time turbulence velocity correlations and the mean shear flow distribution. In the next section the theory is applied to a shear flow and the acoustic power emission is calculated.

11. PREDICTED ACOUSTIC POWER FOR SIMPLE SHEAR FLOW

The sound power intensity equation, Eq. (10.14), requires the spatial distribution of turbulence and shear. The test case selected to evaluate the model developed here is an axisymmetric, annular shear layer. The three-dimensional turbulence in this annular layer was calculated based on a one-dimensional shear. Figures 6 and 7 illustrate the geometrical details. Figure 6 shows a one-dimensional velocity profile schematically with boundary conditions $U(-\infty) = 0$ and $U(+\infty) = U$. The turbulence was calculated using the second-order closure model of Donaldson, with the mean profile allowed to evolve to a self-similar distribution. The shear layer was treated as axisymmetric as shown in Figure 7, where the annular layer is treated as thin, $\Delta_s << R$, thus justifying the one-dimensional mean profile calculation. The core velocity is uniform and equal to U_c .

The resulting turbulence and mean velocity gradient was then used to calculate $I(\psi)$ as given by Eq. (10.14), with $U_m/U_c = \frac{1}{2}$. Since the τ dependence of the decorrelation function b is unknown, the test case provided the opportunity to empirically assign values to the expansion coefficients b_n . This was done by choosing those values of the coefficients which provided agreement between the calculations and an actual sound power intensity measurement in decibels. The test data selected were those reported by Lush.[30] These measurements were carried out using an axisymmetric subsonic jet as the sound source. The annular shear layer assumption of the present analysis precluded a direct comparison of the actual magnitude of sound intensity. Therefore, the coefficients were assigned to provide agreement with the measured ψ dependent directivity. This was done by referencing the intensity to a selected value at $\psi = 0$

$$dB(\psi) - dB(o) = 10 \log_{10}\left(\frac{I(\psi)}{I(o)}\right) \tag{11.1}$$

Figure 8 presents the comparison obtained for the following $b(\tau)$ expansion near $\tau = 0$

$$b(\tau) = 0.02 \, [1 + 16A\tau - 6A^2\tau^2 - 2A^3\tau^3 \, \ldots.] \tag{11.2}$$

i.e., $b_o = 0.02$, and specifying $dB(o) = 100$ for V_j , the measured jet velocity, equal to 300 m/sec. The predicted distributions for V_j = 195 and 125 m/sec. were obtained by decreasing the intensity using the ratio of jet velocities to the eighth power. Note that the predictions contain no convection or refraction effects.

The separate contributions of the self noise and shear noise may be seen in Figure 9. For the values of b_n selected to match the measured directivity the self noise is nine times as large as the shear noise along the jet axis. Although the total sound power distribution is in agreement with measured values, this ratio and the

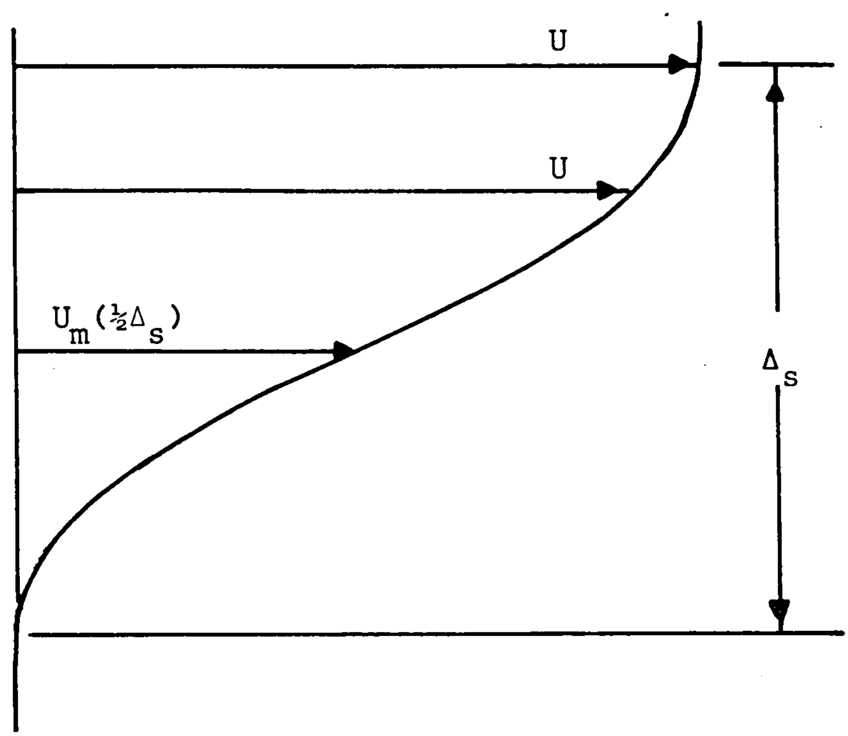


Figure 6. One-dimensional shear layer model

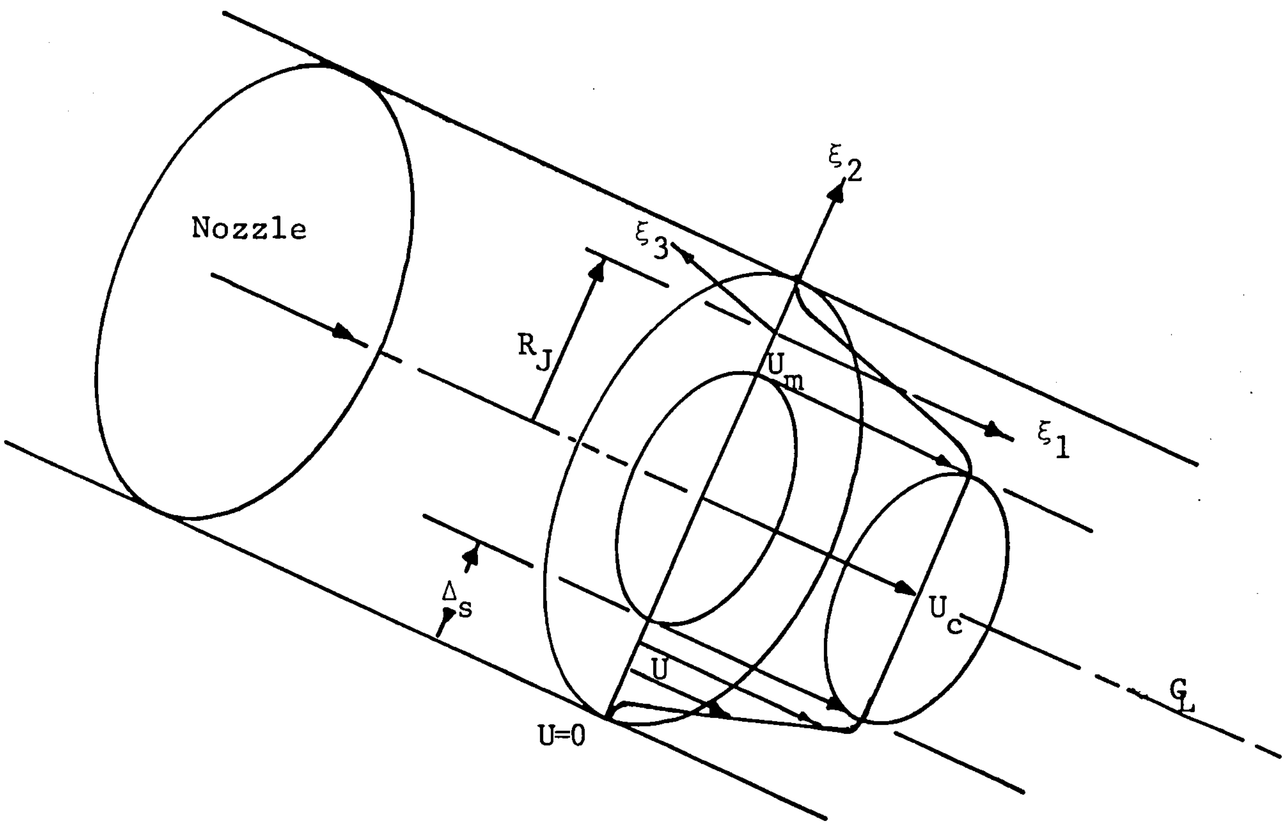


Figure 7. Configuration of axisymmetric flow used for acoustic calculations, with the assumption that $\Delta_s << R_J$. Flow is laminar in the core region and outside the jet

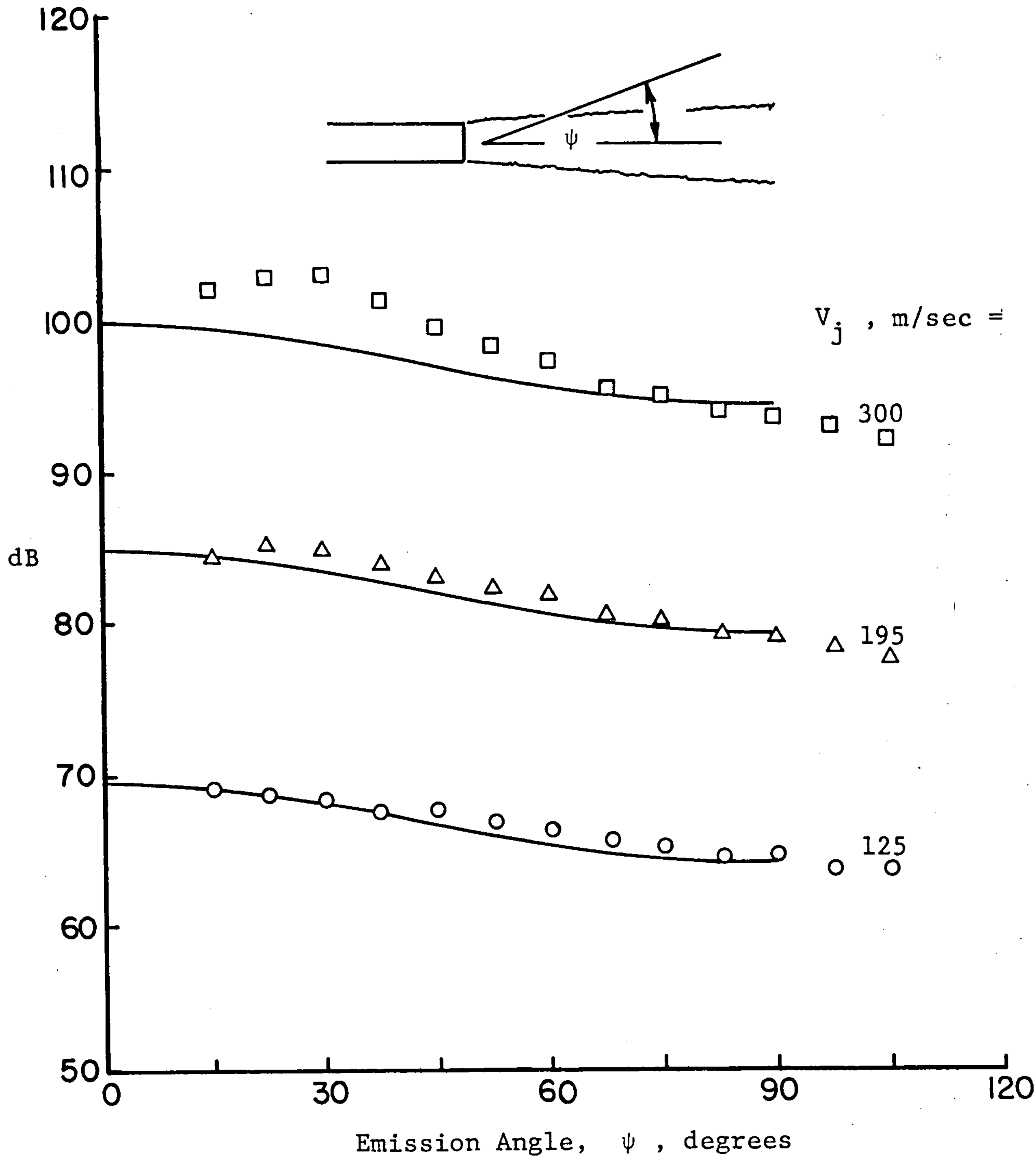


Figure 8. Comparison of theoretical directivity with data of Lush[30]

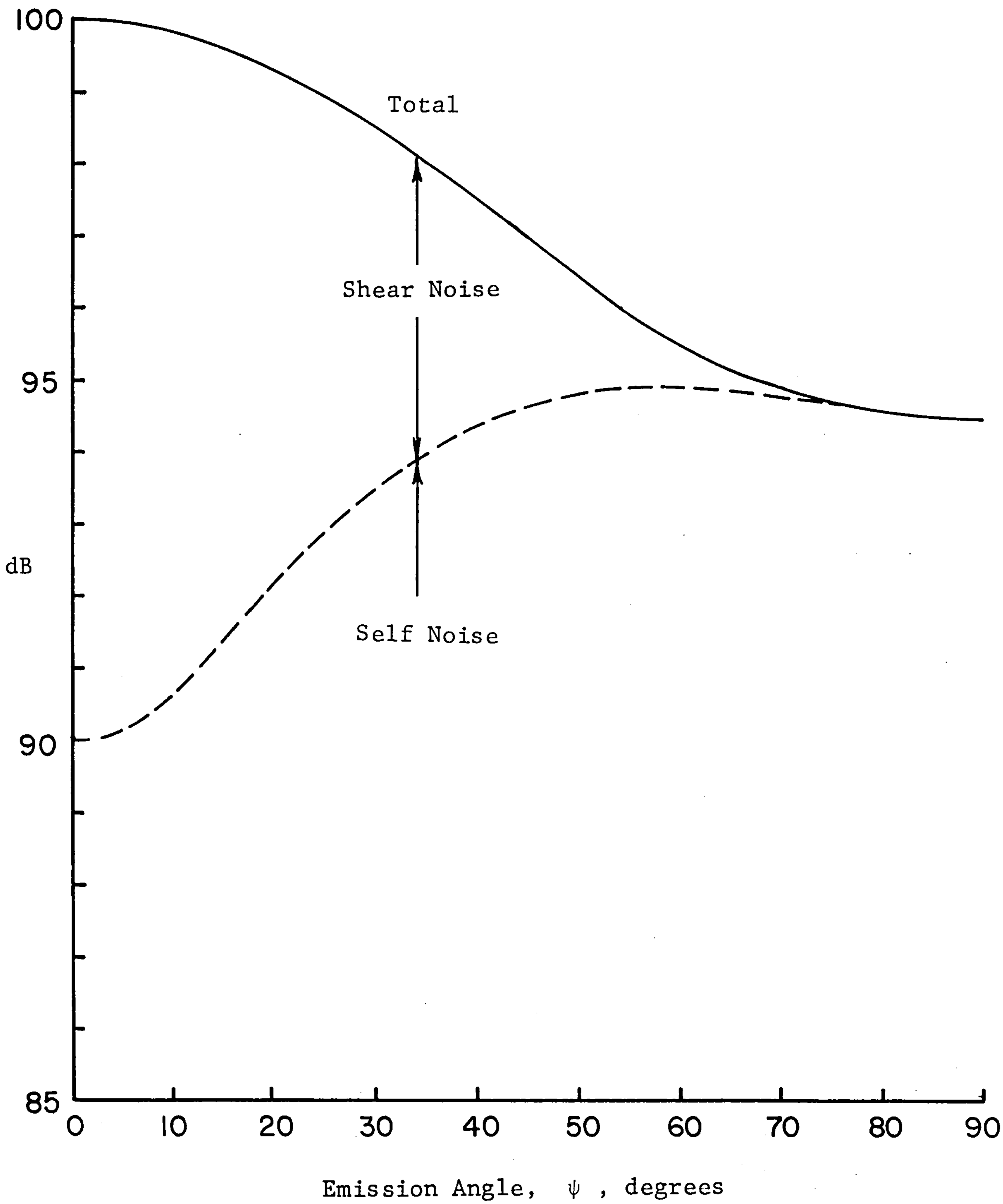


Figure 9. Theoretical directivity including anisotropic turbulence effects

relative contributions of the self and shear noise predicted here are only a preliminary assessment. Several factors which have a large influence on the results require further study. The Q_{13} and Q_{23} correlations have not been included at this point and the value of $b = 0.02$ was chosen arbitrarily. Further effort is required to define the behavior of $b(\tau)$. It should be noted, however, that there appears to be a unique set of coefficients in the expansion of $b(\tau)$ which guarantees a decay in $g_{ij}(\tau)$ (i.e., the second derivative of $g_{ij}(\tau)$ less than zero) for a given choice for $b(\tau)$ and the selected value of $\Lambda_{ij}^{(k)}$, which also correctly predicts the measured directivity distribution. The nondimensional length scale $\Lambda_{11}^{(2)}(0) = 1.0$, calculated using the data of Champagne, et al.[17] was used.

Additionally, the spatial integrals over the jet volume are based on an approximate velocity profile. Finally, as shown by Goldstein and Rosenbaum,[26] the ratios of the correlation scales have a profound effect on the distribution of sound power. The values of the scales for the present results appear to be too large in the flow direction and too small in the direction perpendicular to the mean velocity and the velocity gradient. No information on the integral scales exists for the Lush tests and, therefore, the use of the nondimensional scale length $\Lambda_{11}^{(2)} = 1.0$ was continued. Investigation of the decorrelation function $b(\tau)$ and the influence of the anisotropic scales in the generation of the sound power should be part of the next phase of study.

12. CONCLUSIONS AND RECOMMENDATIONS

The feasibility of computing aerodynamic sound using a new approach for the prediction of Q_{ij} , the two-point, two-time velocity correlations, has been demonstrated for a certain class of flows. The agreement between measured and theoretical sound power emission directivity and the ability of the technique to predict the behavior of Q_{ij} in spatial separation indicates that one-point, one-time turbulence models can be successfully extended to the two-point, two-time problem. The chosen form of the correlation function is vindicated by the favorable comparison between theoretical results and measured turbulent correlations in three directions in separation space. The results provide confidence that the present approach is correct, although further effort is required. Three areas which are recommended as subjects for the next phase of study are: extension of the theory to noncompact sound generation; investigation of the decorrelation function to provide a physical basis for the specified variation of its separation-time behavior; and further consideration of anisotropic scale length effects. An axisymmetric counterpart to the one-dimensional shear layer calculation presented in this report should be developed to permit application of the technique to axisymmetric jets.

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16. Abstract

Calculation of turbulence-generated aerodynamic sound requires knowledge of the spatial and temporal variation of $Q_{ij}(\xi_k,\tau)$, the two-point, two-time turbulent velocity correlations. In this study, a technique is presented to obtain an approximate form of these correlations based on closure of the Reynolds stress equations by modeling of higher-order terms. The governing equations for Q_{ij} are first developed for a general flow. The case of homogeneous, stationary turbulence in a unidirectional constant shear mean flow is then assumed for this initial study.

The required closure models are based on an existing one-point, one-time closure technique. An approximate form for Q_{ij} is selected which is capable of qualitatively reproducing experimentally observed behavior. This form contains separation-time-dependent scale factors as parameters and depends explicitly on spatial separation. An integral technique is used to avoid a full numerical solution in space and time. The approximate forms of Q_{ij} are used in the differential equations and integral moments are taken over the spatial domain. The resulting set of ordinary differential equations are solved in separation time.

The velocity correlations so determined are used in the Lighthill theory of aerodynamic sound by assuming normal joint probability. A preliminary evaluation of the technique is obtained for a specific flow by matching the theoretical result with sound measurements in an axisymmetric jet.

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